Bounds on the cardinality of a topological space

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Abstract: Cardinality bounds on topological spaces have a long and storied history and are very much an active area of research in set-theoretic topology today. A major breakthrough came in 1969 when A.V. Arhangel'skii introduced a fundamentally new technique to establish that the cardinality of a compact, first-countable, Hausdorff space is at most the cardinality of the reals. This answered a question that had been open for 50 years prior. Bounds on topological spaces are expressed in terms of "cardinal invariants"; i.e. cardinal-valued functions that measure a certain property of a space. One such cardinal invariant is the weight of a space, which is the least cardinality of a basis. Another is the density, which is the least cardinality of a dense subset. In this talk we will begin with a little set-theoretic background, survey several important cardinal invariants, and present a few basic cardinality bounds. We will also explore the deeper results of Arhangel'skii and Hajnal-Juhász. Finally, many bounds can be improved if the space is known to be topologically homogeneous; that is, for every pair of points in the space there is an autohomeomorphsim of the space that maps one point to the other. In a homogeneous space, the topology at every point is "identical" to the topology at any other point. In 1978, Erik van Douwen established the first known cardinality bound for homogeneous, Hausdorff spaces. We will give a survey a few results of the speaker in this connection.