Number Theory

Easy (25 Points):

What is the greatest common divisor of 750, 810, and 180?

Medium (50 Points):

Proposition: If n > 1 is a composite number, then there is a prime number p such that p divides n and $p \leq \sqrt{n}$. Using this proposition, determine if 247 is prime or composite.

Hard (100 Points):

There are five sealed boxes before you. One of them contains a chocolate bar. You are told that if you count them back and forth across the line (A, B, C, D, E, D, C, …), then the 2017th box has the chocolate bar. Which box contains your prize?

A	B	C	D	E
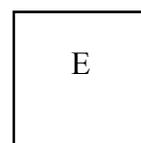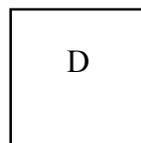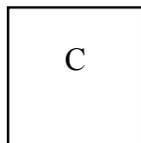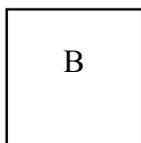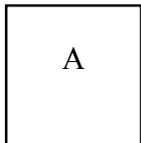
Graph Theory

A graph is a mathematical object that is composed of a set of vertices and a set of edges. A graph can be represented visually as shown. The vertices are depicted by dots. An edge must join two vertices; edges are shown as lines (or arcs) between two dots. The graph shown has six vertices and nine edges. An edge between two vertices represents the existence of a relationship between those two vertices. Two vertices that do not have this relationship are not joined by an edge. Two vertices are called adjacent if there is an edge that joins them.

Here is an example of a visual representation of a graph:
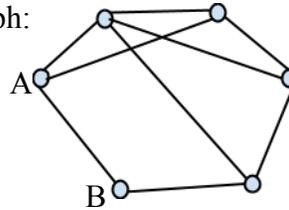
Easy (25 Points): A complete graph with n vertices, K_n, is a graph in which all possible edges between pairs of vertices are included in the graph. Ex: Here is a pictorial representation of K_4:
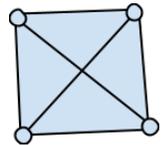
How many edges are present in K_{12}?

Medium (50 Points): The degree of a vertex in a graph is the number of edges incident with ("touching" or "coming out of") the vertex. For example, in the first graph above, the degree of A is 3, while the degree of B is two. If the degrees of the vertices of a graph are: 4, 3, 2, 2, 2, 1, 1, 1, how many edges does the graph have?

Hard (100 Points): After the competition, you decide to take a walk around campus. You want to visit all of the locations below and walk through each path on the map exactly once. If you begin and end your walk at the Science Center, is this possible (using map below)? If yes, list the locations in the order visited.
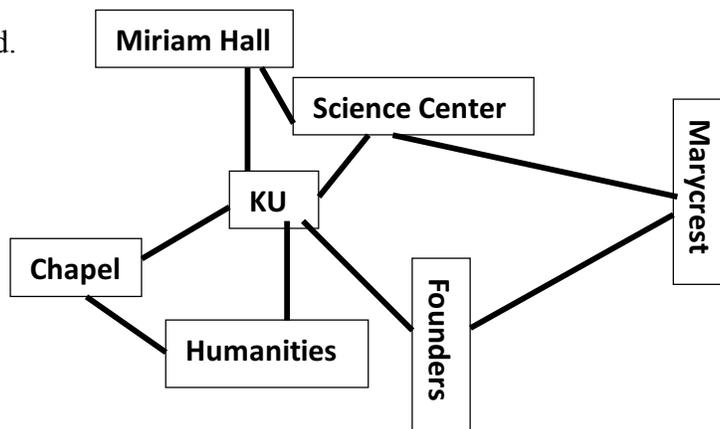
Probability

Easy (25 Points): Fred buys online half the time he shops. There's a 90% chance Fred gets a discount when he orders a TV online. There's a probability of 0.3 that he gets a discount when he buys a TV in store. What's the probability he gets a discount on his TV?

Medium (50 Points): A bag of 15 skittles has 2 purple, 4 green, and 1 yellow. You randomly select two from the bag and eat them immediately. What is the probability that you ate one purple and one that is none of the colors mentioned? Keep your answer as a fraction.

Hard (100 Points): Tina randomly selects two distinct numbers from the set {1, 2, 3}, and Sergio randomly selects a number from the set {1, 2, … 10}. What is the probability that Sergio's number is larger than the sum of the two numbers Tina chose?

Trigonometry

Easy (25 Points): Will is biking up Mirkwook street on his way home and comes to a hill. Will remembers he needs to use inclined surface for an upcoming science project and wonders if he can use the hill. On a map, the hill is 10m long and is said to be at an angle of 60 degrees above the horizon. If Will needs a surface of at least 25m for his project, will the hill be long enough to use?

Medium (50 Points): Two buildings are some distance apart. One of the buildings is 200 meters tall. From the top of the 200 meter building, the top of the opposing building is 10 degrees above the horizontal. Also from the top of the 200 meter building, the bottom of the opposing building is 20 degrees below the horizontal. How tall is the other building?
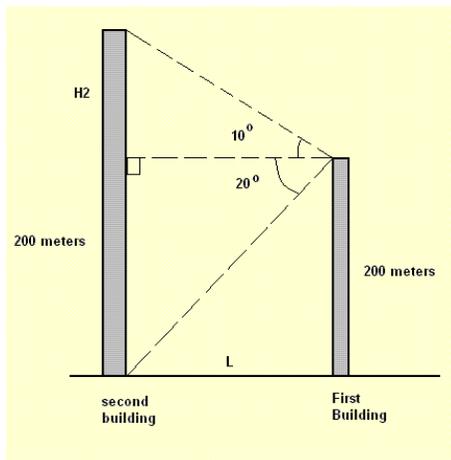
Hard (100 Points): In triangle ABC, tangent of angle CAB is 22/7. Line AD perpendicularly intersects line BC at point D and splits BC into lengths 3 and 17. What is the area of triangle ABC?
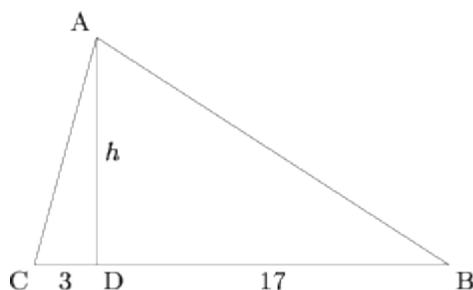
Algebra

Easy (25 Points): Given a function $f(x) = -100x^2 + 100x - 100$ for $-100 \leq x \leq 100$,

what is the maximum value of $|f|$?

Medium (50 Points): Find x, y, z:

$$5x + 3y - 2z = -19$$

$$8x - 6y + 3z = 84$$

$$-23x + 7y + 10z = -31$$

Hard (100 Points): Find value of $\log_y(x^8)$ if $\log_x(y^3)=2$

Geometry

Easy (25 Points):

Find ∠A below, in terms of degrees. Graph is not necessarily drawn to scale.
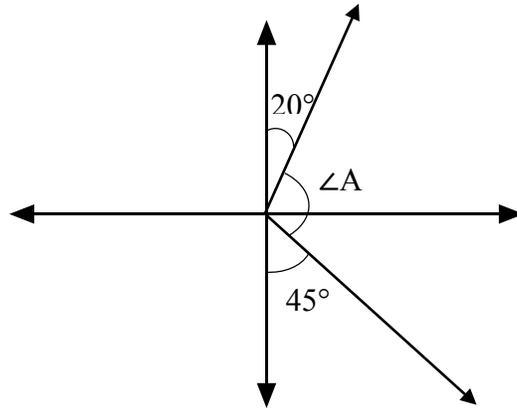
Medium (50 Points):

The circle below has a radius of r =10. Find the exact length of chord a, knowing the given angle of 120°. Figure is not necessarily drawn to scale.
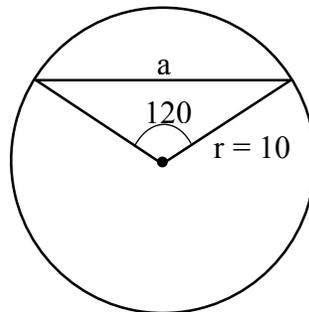
Hard (100 points):

In the figure below, the inner circle inscribes a square. The square is circumscribed by a larger circle. The larger circle inscribes a triangle. The triangle is an equilateral triangle with side length *a*. Find the area of the shaded region, in terms of the constant π and the variable *a*.
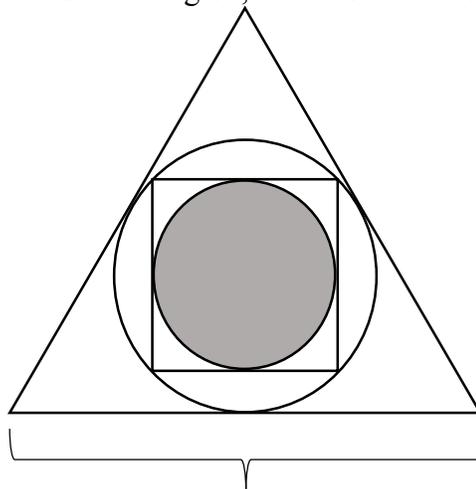
History of Math

Easy (25 points):

In an attempt to "square the circle," the Indiana House of Representatives unanimously passed a bill in 1897 to the Senate that would have implied a value of π of 3.2. The bill was tabled in the Senate thanks to a Purdue University professor. If the diameter of a circle were 100 inches, by how many inches (rounded to the nearest inch) would the circumference of the circle be off if $\pi = 3.2$ were assumed? For the purposes of this problem, assume the actual value of π to be 3.14.

Medium (50 points):

In the decimal numeral system (numbers with base 10),
$2017 = 2017_{dec} = (2017)_{10} = 2 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$.
The octal numeral system consists of numbers with base 8, and is seen in computer science. Convert the octal number $(2017)_8$ to its equivalent number in decimal numeral system (base 10).

Hard (100 points):

Partial fraction decomposition is historically an important step in various types of problems.
Decompose the expression $\dfrac{4x^3+3x^2+2x+1}{(x^2+3x+2)(x^2+7x+12)}$ into its partial fractions.

Surprise

Easy (25 points):

What is equivalent of the following expression in the form a+b*i*?

$$\sqrt{-(4)^3}$$

Medium (50 points):

What does x equal after the execution of this algorithm.

x = 4

while (x < 20){

      if (x is odd)

            multiply x by 2

      else

            Subtract 1 from x

}

Hard (100 points):

At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

Logic

Easy (25 points): Three large glasses of orange juice contain the same amount as four medium glasses. Three medium glasses of juice contain the same amount as four small glasses. How many small glasses contain the same amount as nine large glasses?

Medium (50 points):

Hard (100 points): Weekly Movie Night

Elliot was pleased to note that there were a number of new movies releasing this month. He checked with his friends and several expressed an interest in joining him. Each of his friends wanted to see just one of the movies and they all picked different movies so he had all of his selections covered. They went to see the movies in the evening the week after the movie released. The day of the week varied however depending upon the schedules of the friend joining him. Determine the title and release date of each movie that Elliot went to see (one release date was January 23rd), the day of the week that they went to see each movie, and the name of the friend who went with Elliot each week. Use the table below to help you.

1. "Heavenly Stars" released on January 30th but Elliot didn't see it on a Thursday night. Nathan went with Elliot on a Monday night but they didn't see "Victorious".
2. George didn't join Elliot on a Wednesday night. Elliot saw the movie that released on January 9th on a Friday night but not with Tim.
3. "The Affair" came out a week after the movie that Elliot saw with George but a week before the movie that Elliot saw on a Thursday night.
4. Elliot did not see the movie that released on January 16th on a Tuesday night.
5. Elliot saw "At Midnight" before he went with Brad but two weeks after "Just Before Dawn".
6. Lester went to see the movie that released on January 2nd but not on a Wednesday night.

	January 2nd	January 9th	January 16th	January 23rd	January 30th	Monday	Tuesday	Wednesday	Thursday	Friday	Brad	George	Lester	Nathan	Tim
"At Midnight"															
"Heavenly Stars"															
"Just Before Dawn'															
"The Affair"															
"Victorious"															
Brad															
George															
Lester															
Nathan															
Tim															
Monday															
Tuesday															
Wednesday															
Thursday															
Friday															

Counting

Easy (25 points):

A combination is a way of choosing items from a collection of items such that the order of choosing does not matter. For example, (a,b,c) is the same selection as (c,a,b) if we were selecting a combination of three letters from the alphabet. Given n items to choose from to make groups of size k, we use the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

There are twelve players on the A10 Conference Champion University of Dayton Flyers Women's Basketball team. How many ways can the coach choose a combination of five of the twelve players on the team to start the game? Your answer must be an integer.

Medium (50 points): A magic square is a square array constructed out of the integers $1,2,3,\ldots n^2$ such that the sum of each row, each column, and each of the two diagonals is the same. This sum is called the magic sum of the magic square. An example of a magic square of order 3 with magic sum 15 is

8	1	6
3	5	7
4	9	2

Find the values of A, B, C, D, and E in the following magic square

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	*A*	22	*B*	17	*C*
25	29	10	9	26	12
36	*D*	33	*E*	2	31

Hard (100 points): A Latin square is a square array of numbers or symbols such that no symbol appears twice in the same row or column.

By exchanging the contents in some of the cells in the array below, the letters F, L, Y, E, and R and the symbols +,-,×,÷, and = can each be made into a Latin square.

Note: you must exchange both the letter and the symbol at the same time.   For example, exchanging the contents of cell 1A and cell 1E would make the first row and the first column each contain the correct letters, but would put two equals signs in the first column.

Find the smallest number of cells that need to be exchanged to make both arrays into Latin squares, and list the cells that you need to exchange.

	Col A	Col B	Col C	Col D	Col E
Row 1	+F	-L	×Y	÷E	=R
Row 2	=L	-E	+Y	×R	÷F
Row 3	÷Y	=E	+R	-F	-Y
Row 4	×E	÷R	=F	+L	×L
Row 5	×F	-R	÷L	=Y	+E

Number Theory (answers)

Easy (25 Points): 30

$750 = 2*3*5*5*5$

$810 = 2*3*3*3*3*5$

$180 = 2*2*3*3*5$

common prime factors: $2*3*5 = 30$

Medium (50 Points): 247 is composite.

$\sqrt{247} \cong 15.7$

2, 3, 5, 7, and 11 do not divide 247.

13 divides 247.

As $13 < 15.7$, 247 is composite.

Hard (100 Points): Box A

Counting in the way described creates a length 8 loop (ABCDEDCB) of boxes before restarting

the count at the first box. In this way, we can think of the loop being in modulus 8.

Note, 2017 (mod 8) = 1, so box A is the 2017th box. (In all transparency, this problem is inspired

by the "Seven Goblets" word problem on Brilliant.org's page about modular arithmetic.)

Graph Theory (answers)

Easy (25 Points): $\frac{n(n-1)}{2} = \frac{12*11}{2} = \frac{132}{2} = 66$

Maximum number of edges is $\binom{n}{2}$.

Medium (50 Points): 8 edges

The sum of the degrees of the vertices of a graph is equal to twice the number of edges in the graph. As, $4 + 3 + 2 + 2 + 2 + 1 + 1 + 1 = 16$, the number of edges is 8.

Hard (100 Points): Not possible

Multiple vertices with odd degree

Probability (answers)

Easy (25 Points): 0.6 or 3/5

$(0.5 * 0.9) + (0.5 * 0.3) = .45 + .15 = 0.6$.

Medium (50 Points): 16/105

$2 * 2/15 * [15 - 2 - 4 - 1]/14 = (4/15 * 8/14) = 32/210 = 16/105$

Hard (100 Points): 3/5 or 0.6

Tina can get a sum of 3 one way, choosing (1, 2). Sergio can choose a number 4 – 10, so 7 ways.

Getting a sum of 4 one way, choosing (1, 3). Sergio can choose number 5 – 10, so 6 ways.

Get a sum of 5 one way, choosing (2, 3). Sergio can choose 6 – 10, so 5 ways.

Total ways Sergio can pick is $7 + 6 + 5 = 18$. Tina chooses two distinct numbers in 3 ways, and

Sergio can in 10 ways, making $3*10 = 30$. Therefore, probability $= 18/30 = 3/5$.   (or 0.6)

Trigonometry (answers)

Easy (25 Points): No. Too Short

The hill can be represented as a triangle with a base of 10m and an angle of 60 degrees. The distance of the hill should be equal to the length of the hypotenuse or $\frac{10}{\cos(60)}$ due to $\cos(60) =$ $\frac{10}{Hyp.}$. Because cosine of 60 is 0.5, the length of the hypotenuse is 20m. Thus, the hill is two short for Will's experiment.

Medium (50 Points): 200 + 200 * tan(10)/tan(20)

tan(20) = 200 / L

L = 200 / tan(20)

tan(10) = H2 / L

H2 = L * tan(10)

= 200 * tan(10) / tan(20)

Height of second building = 200 + 200 * tan(10) / tan(20)

Hard (100 Points): 110

Given the triangle, $\tan DAB = \frac{17}{h}$ and $\tan CAD = \frac{3}{h}$. Then:

Addition of angles	$\tan CAB = \tan(DAB + CAD)$
Tangent sum formula	$\frac{22}{7} = \frac{\tan DAB + \tan CAD}{1 - \tan DAB \cdot \tan CAD}$
Simplify	$= \frac{\frac{17}{h} + \frac{3}{h}}{1 - \left(\frac{17}{h}\right)\left(\frac{3}{h}\right)}$
Cross Multiplication	$\frac{22}{7} = \frac{20h}{h^2 - 51}$
Divide by 2 and factor	$0 = 22h^2 - 140h - 22 \cdot 51$
	$0 = (11h + 51)(h - 11)$

Because h can't be negative h = 11, thus making the area: $\frac{1}{2}(17 + 3) * 11 = 110$

Algebra (answers)

Easy (25 Points): 1,010,100

Plug in $x = -100$.

Medium (50 Points): $x = 3, \; y = -6, \; z = 8$

Hard (100 Points): 12

x^2=y^3 by log definition.

x^8=y^12 by taking both sides to the fourth power. Plug in y^12 for x^8 in the first log expression.

logy(y^12)=12 By the power rule.

Geometry (answers)

Easy (25 Points):

115°

Explanation:

The angles protrude from the origin to the right of the y axis, and a straight line (the y axis in this case) is 180°. So, the angles being considered must add up to 180°.
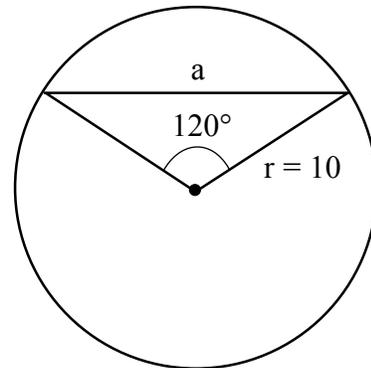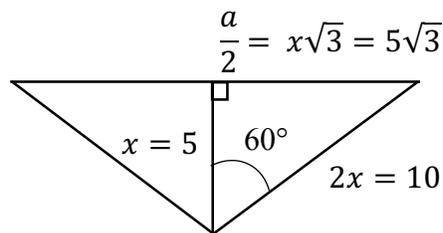
$$45° + \angle A + 20° = 180°$$

$$\angle A = 115°$$

Medium (50 Points):

$10\sqrt{3}$

Explanation:

Triangle is split in half (by angle bisector with forms right triangle) and becomes a 30°-60°-90° triangle. Such a triangle is known to always have side lengths of $x, x\sqrt{3}, and\ 2x$. In the figure below and to the left, 2x = r = 10. Therefore, a/2 = $5\sqrt{3}$, and a = $10\sqrt{3}$.

$$\frac{a}{2} = x\sqrt{3} = 5\sqrt{3}$$

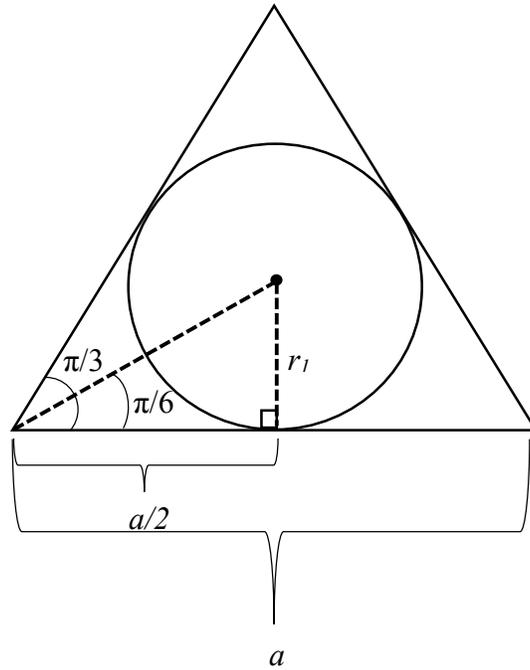$x = 5$ 60°

$2x = 10$

a

120°

r = 10

Hard (100 Points):

$$\frac{\pi}{24}a^2$$

or equivalent.

Explanation:

$$\tan\frac{\pi}{6} = \frac{r1}{\left(\frac{a}{2}\right)}$$

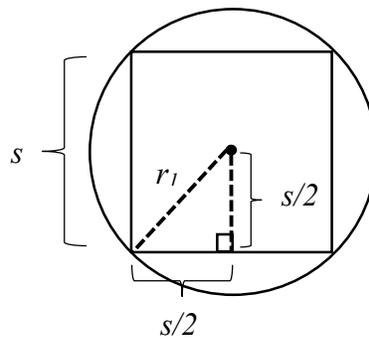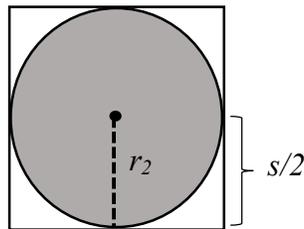$$r_1 = \frac{\sqrt{3}}{6}a$$

$a/2$

a

$$(r_1)^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2$$
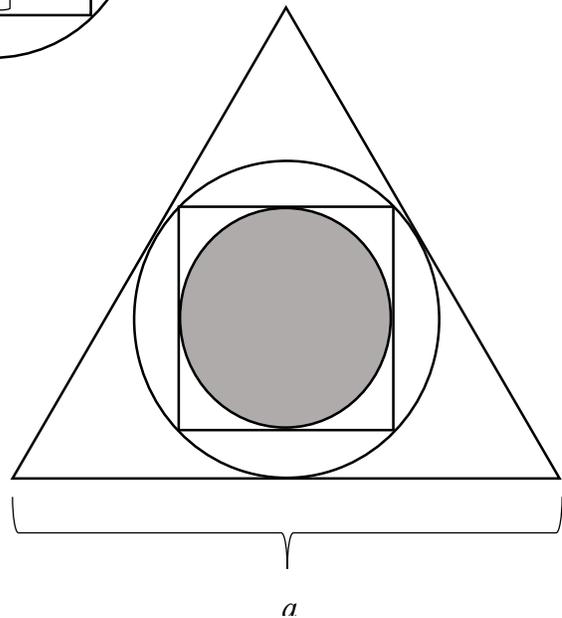
$$s = \sqrt{2}\,r_1$$

$$s = \sqrt{2}\left(\frac{\sqrt{3}}{6}a\right) = \frac{\sqrt{6}}{6}a = \frac{1}{\sqrt{6}}a$$

s

r_1

$s/2$

$s/2$

$$r_2 = \frac{s}{2} = \frac{1}{2\sqrt{6}}a$$

r_2

$s/2$

$$A_{shaded\ region} = \pi(r_2)^2 = \frac{\pi}{4\times6}a^2 = \frac{\pi}{24}a^2$$

a

History of Math (answers)

Easy (25 points):

6 inches

Explanation:

circumference $= \pi \times$ diameter

circ. (accurate) $= \pi \times 100$ inches ≈ 314 inches

circ. (inaccurate) $= 3.2 \times 100$ inches $= 320$ inches

circ. (inaccurate) – circ. (accurate) ≈ 6 inches

Medium (50 points):

1039 or *(1039)₁₀* or *1039$_{dec}$* or equivalent.

Explanation:

$$(2017)_8 = 2{\times}8^3 + 0{\times}8^2 + 1{\times}8^1 + 7{\times}8^0 = (1039)_{10}$$

Hard (100 points):

$$-\frac{1}{3(x+1)} + \frac{23}{2(x+2)} - \frac{43}{x+3} + \frac{215}{6(x+4)}$$

or equivalent.

Explanation:

$$\frac{4x^3+3x^2+2x+1}{(x^2+3x+2)(x^2+7x+12)} = \frac{4x^3+3x^2+2x+1}{(x+1)(x+2)(x+3)(x+4)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} + \frac{D}{x+4}$$

Fastest way: use Heaviside cover-up method.

$$A = \left.\frac{4x^3+3x^2+2x+1}{(x+2)(x+3)(x+4)}\right|_{x=-1} = -\frac{1}{3}$$

$$B = \left.\frac{4x^3+3x^2+2x+1}{(x+1)(x+3)(x+4)}\right|_{x=-2} = \frac{23}{2}$$

$$C = \left.\frac{4x^3+3x^2+2x+1}{(x+1)(x+2)(x+4)}\right|_{x=-3} = -43$$

$$D = \left.\frac{4x^3+3x^2+2x+1}{(x+1)(x+2)(x+3)}\right|_{x=-4} = \frac{215}{6}$$

Replace variables with their respective values.

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} + \frac{D}{x+4} = -\frac{1}{3(x+1)} + \frac{23}{2(x+2)} - \frac{43}{x+3} + \frac{215}{6(x+4)}$$

(Traditional method also possible.)

Surprise (answers)

Easy (25 points): 8 i

Medium (50 points): x = 34

Hard (100 points): 245

Logic (answers)

Easy (25 points): 16

Medium (50 points):

Hard (100 points):

Movie Title	Release Date	Day of the Week	Friend's Name
"At Midnight"	January 23rd	Thursday	Tim
"Heavenly Stars"	January 30th	Wednesday	Brad
"Just Before Dawn"	January 9th	Friday	George
"The Affair"	January 16th	Monday	Nathan
"Victorious"	January 2nd	Tuesday	Lester

Counting (answers)

Easy (25 points): $\left(\frac{12!}{5!7!}\right) = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 792$

Medium (50 points): A = 20, B = 21, C = 13, D = 5, E = 4

Hard (100 points): 3 exchanges:

1. 2B and 2C switched
2. 3E and 4E switched
3. 5A and 5B switched

Correct matrix below

+F	-L	×Y	÷E	=R
=L	+Y	-E	×R	÷F
÷Y	=E	+R	-F	×L
×E	÷R	=F	+L	-Y
-R	×F	÷L	=Y	+E