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Recombining Tree for Regime-Switching Model: Algorithm and Weak Convergence

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This paper is concerned with a multinomial tree method for option pricing when the underlying asset price follows a regime-switching model. A direct extension of the well-known CRR tree to the regime-switching model is examined first. Then an efficient recombining tree is presented that grows linearly as the number of time steps increases. The weak convergence of the discrete multinomial approximations to the continuous regime-switching process is proved via the method of martingale problem formulation.

Key Words. Regime-switching model, multinomial tree, option pricing, weak convergence.

1. Introduction

Lattice methods have been broadly adapted in computational finance ever since the innovative work by Cox, Ross and Rubinstein (Ref. 7) for the Black-Scholes-Merton option pricing model (the CRR binomial tree). See Refs. 2,6,8,10,14–16,21–23 and the references therein. From the computational perspective, a crucial feature for a successful tree design is that the number of nodes can not grow too fast as the number of time steps increases. According to Nelson and Ramaswamy (Ref. 21), a binomial approximation to a diffusion is "computationally simple" if the number of nodes grows at most linearly in the number of time intervals. One of the main reasons for the CRR method being practically popular is because it grows linearly and is therefore computationally simple.

Pricing derivative securities in regime-switching model has drawn considerable attention in recent decade. See Refs. 1,3–5,9,11–13,17,20,24, among others. In this setting, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents random switch among different regimes. Model parameters (drift and volatility coefficients) are assumed to depend on the Markov chain. In certain cases, closed-form solutions can be obtained for option prices. For instance, Guo (Ref. 11) provided an analytical formula for the European call option prices assuming there are two regimes; Buffington and Elliott (Ref. 5) provided a general formula for $m \geq 2$ regimes; Guo and Zhang (Ref. 12) studied a perpetual American put option with two regimes and derived an analytical solution. However, it is extremely difficult to obtain a closed-form solution for many other cases, in particular, the American options with finite expiration time and with $m \geq 2$ regimes. Hence seeking efficient numerical approximation schemes comes up as an important alternative.

In Ref. 19, the author has developed a multinomial tree approach for regime-switching models which grows linearly as the number of time steps increases. This recombining tree allows us to use large number of time steps to obtain accurate approximations for both European and American option prices. We have also developed an approximation regime-switching model for the well-known Heston's stochastic volatility model and then employed the tree approach for pricing options. It is noted (see, for example Ref. 28) that using a regime-switching model makes it possible to describe stochastic volatility in a relatively simple manner (simpler than the so-called stochastic volatility models). Various numerical examples are studied in Ref. 19. See Ref. 19 for those numerical results and further discussions of the method.

The present paper is concerned with the weak convergence of the multinomial tree approximations to the continuous regime-switching diffusion process. For models without regime-switching, results on convergence of the binomial tree approximations have been obtained in a number of articles. See Refs. 6,14,15,21,23 and the references therein. However, due to the added Markov chain in the underlying model, the methods used in these papers can not be directly applied to our regime-switching case. Instead, we employ the martingale problem formulation procedure (Ref. 25) to establish our convergence result. This method is developed by Yin, Song and Zhang in Ref. 25 for a class of discrete numerical approximations for diffusions with regime-switching. Also see Ref. 28 for detailed accounts and recent generalizations of the method.

The rest of the paper is organized as follows. Section 2 is concerned with the tree construction. The regime-switching model is introduced in subsection 2.1. A direct extension of the CRR tree to the regime-switching model is examined in subsection 2.2 which is seen not computationally simple. Our new recombining tree is presented in subsection 2.3. Section 3 is devoted to the weak convergence of the tree approximations. We establish the convergence result under a general framework that includes both the completely recombining tree and the partially recombining CRR tree as special cases. Section 4 provides further remarks and concludes the paper.

2. Regime-Switching Trees and Option Pricing

2.1. *Regime-Switching Model*

Let α_t be a continuous-time Markov chain taking values among m_0 different states, where m_0 is the total number of states considered for the economy. Each state represents a particular regime and is labeled by an integer i between 1 and m_0. Hence the state space of α_t is given by $\mathcal{M} := \{1, \dots, m_0\}$. For example, if $m_0 = 2$ (two regimes), then $\alpha_t = 1$ can indicate a bullish market and $\alpha_t = 2$ a bearish market. Let matrix $Q = (q_{ij})_{m_0 \times m_0}$ denote the generator of α_t. From Markov chain theory (see for example, Yin and Zhang (Ref. 26)), the entries q_{ij} in matrix Q satisfy: (I) $q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$ for each $i = 1, \dots, m_0$.

We assume that the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given. Let B_t be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and assume it is independent of the Markov chain α_t. We consider the following regime-switching geometric Brownian motion for the risk-neutral process of the underlying asset price S_t:

$$\frac{dS_t}{S_t} = (r_{\alpha_t} - d_{\alpha_t})dt + \sigma_{\alpha_t} dB_t, \quad t \geq 0, \tag{1}$$

where r_{α_t} is the instantaneous risk-free interest rate, d_{α_t} and σ_{α_t} are dividend rate and volatility rate of the asset, respectively. Note that those parameters depend on the Markov chain α_t, indicating that they can take different values in different regimes. We assume that $\sigma_i > 0$, for each $i \in \mathcal{M}$.

2.2. *A Direct Extension of CRR Tree to Regime-Switching*

A direct extension of the CRR tree (Ref. 7) to the regime-switching model (1) proceeds as follows. See Refs. 1,17.

Let $T > 0$ denote the maturity of option under consideration. Divide the interval $[0, T]$ into N steps. Thus the time step size is given by $h = \frac{T}{N}$. Consider the joint Markov process (S_t, α_t), $0 \leq t \leq T$ and let $(S_k, \alpha_k) := (S_t, \alpha_t)_{t=kh}$ be the state at the kth step of the tree,

$k = 0, 1, \ldots, N$. Assuming $(S_k, \alpha_k) = (S, i)$. Then, at the $(k+1)$th step, the asset price S_{k+1} can either move up to Su_i with probability p_i or move down to Su_i^{-1} with probability $1 - p_i$, where

$$u_i = e^{\sigma_i \sqrt{h}}, \quad p_i = \frac{e^{r_i h} - e^{-\sigma_i \sqrt{h}}}{e^{\sigma_i \sqrt{h}} - e^{-\sigma_i \sqrt{h}}}.$$

On the other hand, the Markov chain α_{k+1} at the $(k+1)$th step may stay at state i with probability p_{ii}^{α} or jump to any other state $j \neq i$ with probability p_{ij}^{α}, where the one-step transition probabilities p_{ij}^{α} of the Markov chain α_k are defined by

$$p_{ij}^{\alpha} = P\{\alpha_{k+1} = j \big| \alpha_k = i\}, \;\; 1 \leq i, j \leq m_0. \tag{2}$$

It then follows that at the $(k+1)$th step, there are totally $2m_0$ possible states for (S_{k+1}, α_{k+1}), given by

$$(S_{k+1}, \alpha_{k+1}) = \begin{cases} (Su_i, i), & \text{with probability } p_i p_{ii}^{\alpha}, \\ (Su_i^{-1}, i), & \text{with probability } (1-p_i) p_{ii}^{\alpha}, \\ (Su_i, j), & \text{with probability } p_i p_{ij}^{\alpha}, \; j \neq i, \\ (Su_i^{-1}, j), & \text{with probability } (1-p_i) p_{ij}^{\alpha}, \; j \neq i. \end{cases} \tag{3}$$

An illustrative two-regime tree is depicted in Fig. 1.

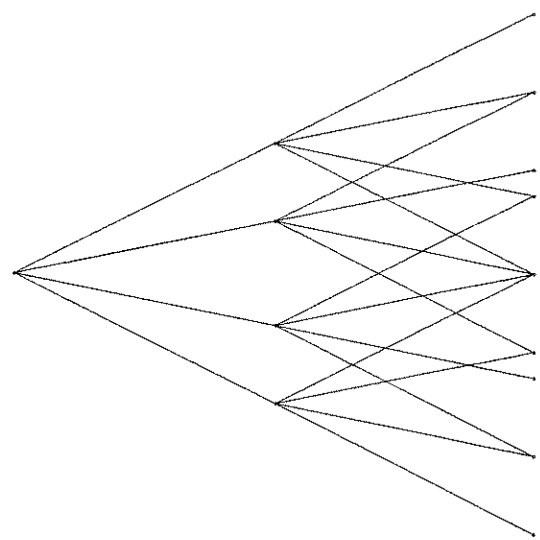

Fig. 1 A non-completely recombining tree for two regimes.

It can be seen that the state space for the two-dimensional Markov chain $\{(S_k, \alpha_k)\}_{k\geq 0}$ is the set

$$\Big\{(S_0 u_1^{2l_1-k_1} u_2^{2l_2-k_2} \cdots u_{m_0}^{2l_{m_0}-k_{m_0}}, i)\Big| 0 \leq l_i \leq k_i, 1 \leq i \leq m_0, \sum_{i=1}^{m_0} k_i = k, 0 \leq k \leq N\Big\}, \tag{4}$$

where S_0 is the initial stock price. It was shown in Aingworth et al. [1, Proposition 1] that the number of distinct asset prices at the kth period is $\binom{k+2m_0-1}{2m_0-1} = O(k^{2m_0-1})$. Thus the tree does not grow linearly and therefore is not computationally simple according to Ref. 21. Consequently, it works well for small number of regimes and for moderate number of tree steps. However, if m_0 and N are large, such an implementation will become computationally formidable due to node explosion. For example, consider a model with 20 regimes. If 100 time steps were used, then the last step of the tree would have $\binom{139}{39} \approx 4.7 \times 10^{192}$ nodes. Apparently. most computers can not handle such a heavy computation requirement.

2.3. *A Regime-Switching Recombining Tree*

To have a tree structure that grows linearly, complete recombination of nodes needs to be achieved at each time step. For the generalized CRR tree discussed in the previous section, node recombination is only achieved partially, namely, nodes do combine if the regime stays the same. However, whenever a switch of regime occurs, the change of asset price will very likely produce new nodes, due to different change factor u_i in different regime. To resolve this issue, we adjust the change factor u_i in a suitable way. Meanwhile, it is necessary to match the local mean and variance calculated from the tree to that implied by the continuous regime-switching diffusion, in order for the discrete tree approximations to converge to its underlying continuous process (see next section). We present the tree design in this section. For numerical experiments and an application of the tree method to the Heston's stochastic volatility model, we refer the reader to Ref. 19.

First, let $X_t = \ln(S_t/S_0)$. Then $S_t = S_0 e^{X_t}$ where X_t is the solution of the stochastic differential equation:

$$dX_t = a_{\alpha_t} dt + \sigma_{\alpha_t} dB_t\,, \quad X_0 = 0, \tag{5}$$

where

$$a_{\alpha_t} = r_{\alpha_t} - d_{\alpha_t} - \frac{1}{2}\sigma^2_{\alpha_t}. \tag{6}$$

We design a completely recombined tree for the process X_t instead of the asset price S_t as done in Refs. 1,3. The corresponding approximations of asset price can be easily obtained from the approximations of X_t. Divide the option life $[0,T]$ into N steps and let $h = \frac{T}{N}$ be the step size. Let $(X_k, \alpha_k) := (X_t, \alpha_t)_{t=kh}$ be the state at the kth step of the tree, $k = 0, 1, \ldots, N$. We use three branches for each regime, a up move, a down move, and a middle stay (no move). The up and down moves are carefully chosen so that they must take values among $2b+1$ evenly spaced points, where the specification of b will be discussed shortly. Let constant $\bar{\sigma} > 0$ and positive integer b be given. Assume $X_k = x$, then at the next step, X_{k+1} must take the three values from the set of $2b+1$ values given by $\{x + j\bar{\sigma}\sqrt{h}, j = -b, -b+1, \ldots, 0, \ldots, b-1, b\}$. Specifically, for regime i at step k, i.e., $(X_k, \alpha_k) = (x, i)$, let l_i (to be determined) be the number of upward moves of X_{k+1}. Then, the three branches associated with regime i are given by $x + l_i\bar{\sigma}\sqrt{h}$, x and $x - l_i\bar{\sigma}\sqrt{h}$. Next, let $p_{i,u}$, $p_{i,m}$ and $p_{i,d}$ be the conditional probabilities corresponding to the up, middle and down branches. By matching the mean and variance implied by the trinomial lattice to that implied by the SDE (5), we have,

$$\begin{cases} (l_i\bar{\sigma}\sqrt{h})p_{i,u} - (l_i\bar{\sigma}\sqrt{h})p_{i,d} = a_i h, \\ (l_i\bar{\sigma}\sqrt{h})^2 p_{i,u} + (l_i\bar{\sigma}\sqrt{h})^2 p_{i,d} = \sigma_i^2 h + a_i^2 h^2, \\ p_{i,u} + p_{i,m} + p_{i,d} = 1, \end{cases} \tag{7}$$

where $a_i = r_i - d_i - \dfrac{1}{2}\sigma_i^2$. It follows that,

$$\begin{aligned} p_{i,u} &= \frac{\sigma_i^2 + a_i(l_i\bar{\sigma})\sqrt{h} + a_i^2 h}{2(l_i\bar{\sigma})^2}, \\ p_{i,d} &= \frac{\sigma_i^2 - a_i(l_i\bar{\sigma})\sqrt{h} + a_i^2 h}{2(l_i\bar{\sigma})^2}, \\ p_{i,m} &= 1 - \frac{\sigma_i^2 + a_i^2 h}{(l_i\bar{\sigma})^2}. \end{aligned} \tag{8}$$

Consequently, emanating from the node (x, i), there are $3m_0$ states for (X_{k+1}, α_{k+1}), given by

$$(X_{k+1}, \alpha_{k+1}) = \begin{cases} (x + l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p_{ij}^{\alpha} p_{i,u}, \\ (x, j) & \text{with probability } p_{ij}^{\alpha} p_{i,m}, \; j = 1, \ldots m_0, \\ (x - l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p_{ij}^{\alpha} p_{i,d}, \end{cases} \tag{9}$$

where p_{ij}^{α} denote the one-step transition probabilities of the Markov chain α_k from state i to j (see (2)). It can be seen that the number of nodes at the kth step is $m_0(2bk+1)$, linear in k. Hence, our design does yield a computationally simple tree. If we consider again the previous example, namely, $m_0 = 20$ and $N = 100$, then the last step of the new tree will have $20(200b+1) = 80020$ if $b = 20$ is chosen, a much smaller number comparing to 4.7×10^{192}.

Fig. 2 displays a recombining tree with 7 branches ($b = 3$) emanating from each node (a heptanomial tree structure).

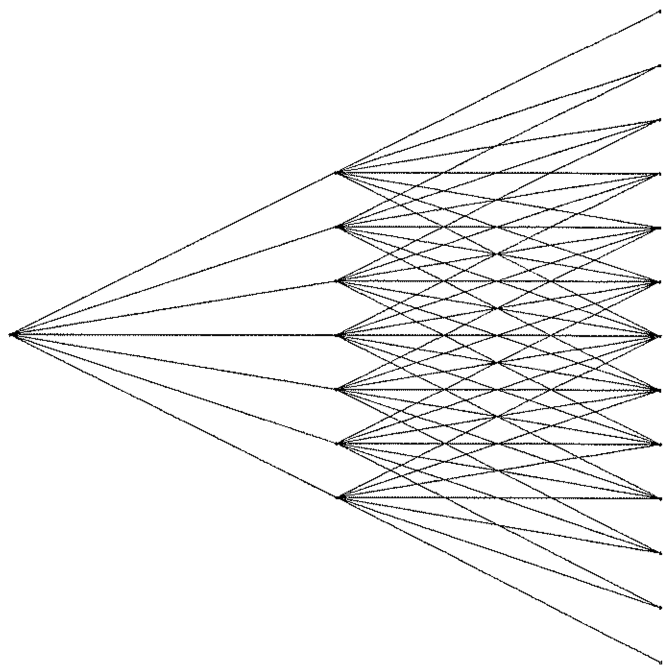

Fig. 2 A two-step recombining heptanomial tree

Note that if l_i and h are not chosen properly, it is possible that (8) results in negative branch probabilities. The following proposition gives conditions under which the branch probabilities are guaranteed to be non-negative.

Proposition 1. *The following assertions hold for the solutions given by (8).*

(1) If $a_i = 0$*, then* $0 \le p_{i,u}, p_{i,m}, p_{i,d} \le 1$ *provided that* $l_i\bar{\sigma} \ge \sigma_i$*;*
(2) If $a_i \neq 0$*, then* $0 \le p_{i,u}, p_{i,m}, p_{i,d} \le 1$ *provided that (a)* $\sigma_i < l_i\bar{\sigma} \le 2\sigma_i$ *and* $h \le \frac{(l_i\bar{\sigma})^2 - \sigma_i^2}{a_i^2}$*, or (b)* $l_i\bar{\sigma} > 2\sigma_i$ *and* $h \le \frac{[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - 4\sigma_i^2}\,]^2}{4a_i^2}$.

Remark 1. The numerical results in Ref. 19 show that the approximate option prices are insensitive to the choice of parameter $\bar{\sigma}$. Hence the choice of l_i can be flexible as long as the conditions in Proposition 1 are satisfied. One way is to choose the l_i value so that the upper constraint for h takes the larger one. Specifically, for given $\bar{\sigma}$, let $k_1 = \lfloor \frac{2\sigma_i}{\bar{\sigma}} \rfloor$ and $k_2 = \lceil \frac{2\sigma_i}{\bar{\sigma}} \rceil$ be the floor and ceiling of $\frac{2\sigma_i}{\bar{\sigma}}$, respectively. Then,

- if either $k_1 = k_2$ or $k_1\bar{\sigma} < \sigma_i$, then set $l_i = k_2$; otherwise,
- let $a = \frac{(k_1\bar{\sigma})^2 - \sigma_i^2}{a_i^2}$, $b = \frac{[k_2\bar{\sigma} - \sqrt{(k_2\bar{\sigma})^2 - 4\sigma_i^2}\,]^2}{4a_i^2}$, if $a \le b$, then set $l_i = k_2$; if $a > b$, then set $l_i = k_1$.

Using the recombining tree we have constructed for the underlying asset process X_t, both European and American options can be priced following a similar procedure that is used in the CRR tree for valuing options in the Black-Scholes model, i.e., by starting at the last step of the tree ($n = N$) and working backward iteratively. At each step and for each node, we first calculate the probability weighted average of all option prices at the nodes in the next step that are directly emanated from the current node. We then discount the averaged future price using the interest rate of the current node. This gives us the discounted expectation of the future option value, where the expectation is taken with respect to the risk-neutral probability $\widetilde{P}$. For American options, in addition we need to check the early exercise possibility by comparing this value with the immediate exercise value. Because of the presence of regime-switching, those computations become more involved and special care needs to take. Note that the option values are functions of the underlying asset price and also depend on the regime. Consider a put option, let $P^n(x,i)$ denote the option value at step n for the node associated with the state $(X_n, \alpha_n) = (x, i)$. Then, for the terminal step $n = N$,

$$P^N(x,i) = \max\{K - S_0e^x, 0\}, \quad i = 1, \dots, m_0, \tag{10}$$

where K is the strike price of the option. At step $n < N$, for European put,

$$\begin{aligned} P^n(x,i) = e^{-r_ih} \sum_{j=1}^{m_0} p_{ij}^{\alpha} \Big[& P^{n+1}(x + l_i\bar{\sigma}\sqrt{h}, j)p_{i,u} \\ & + P^{n+1}(x - l_i\bar{\sigma}\sqrt{h}, j)p_{i,d} + P^{n+1}(x,j)p_{i,m} \Big], \end{aligned} \tag{11}$$

and for American put,

$$P^n(x,i) = \max\left\{K - S_0e^x, e^{-r_ih}\sum_{j=1}^{m_0} p_{ij}^{\alpha}\Big[P^{n+1}(x + l_i\bar{\sigma}\sqrt{h}, j)p_{i,u} + P^{n+1}(x - l_i\bar{\sigma}\sqrt{h}, j)p_{i,d} + P^{n+1}(x, j)p_{i,m}\Big]\right\}. \tag{12}$$

Formulae for call options are obtained by changing $K - S_0e^x$ to $S_0e^x - K$ in (10)-(12).

3. Convergence of the Tree Approximations

In this section we establish the convergence results of the discrete approximation processes implied by the regime-switching trees to the continuous-time processes. We prove the convergence under a general framework that includes the completely recombining tree and the partially recombining CRR tree as two special cases.

We consider the following one-dimensional stochastic differential equation modulated by the Markov chain α_t,

$$dX_t = \mu(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t, \quad t \geq 0, \tag{13}$$

where $\mu(\cdot,\cdot) : \mathbb{R}\times\mathcal{M} \to \mathbb{R}$ and $\sigma(\cdot,\cdot) : \mathbb{R}\times\mathcal{M} \to \mathbb{R}$ are appropriate functions satisfying the Lipschitz condition

$$|\mu(x_1,i) - \mu(x_2,i)| \leq K|x_1 - x_2|, \quad |\sigma(x_1,i) - \sigma(x_2,i)| \leq K|x_1 - x_2|, \tag{14}$$

and the linear growth condition,

$$|\mu(x,i)| \leq K(1 + |x|), \quad |\sigma(x,i)| \leq K(1 + |x|), \tag{15}$$

for all $x, x_1, x_2 \in \mathbb{R}$ and for each $i \in \mathcal{M}$, where K is a positive constant (In what follows we will use K as a generic positive constant; that is, its values may vary at different places). In addition, we assume $\sigma(x,i) > 0$ for all $x \in \mathbb{R}$ and for each $i \in \mathcal{M}$. Note that for the regime-switching model (5) on which the tree construction is based, $\mu(x,i) = r_i - d_i - \frac{1}{2}\sigma_i^2$ and $\sigma(x,i) = \sigma_i$. Clearly, (14) and (15) are satisfied.

For each $i \in \mathcal{M}$, let function $V(x,i)$ be twice continuously differentiable in x. Define an operator $\mathcal{L}$ by

$$\mathcal{L}V(x,i) = \mu(x,i)\frac{dV}{dx}(x,i) + \frac{1}{2}\sigma^2(x,i)\frac{d^2V}{dx^2}(x,i) + QV(x,\cdot)(i) \tag{16}$$

where

$$QV(x,\cdot)(i) = \sum_{j\neq i} q_{ij}[V(x,j) - V(x,i)] = \sum_{j=1}^{m_0} q_{ij}V(x,j) \tag{17}$$

and $Q = (q_{ij})_{m_0 \times m_0}$ is the generator of α_t.

In what follows we will use $\varepsilon > 0$ (instead of h as in the previous section) for the time step size. Then $N = \lfloor \frac{T}{\varepsilon} \rfloor$ is the number of steps for a tree construction. Let the initial state X_0 and initial regime α_0 be given. For each fixed $\varepsilon > 0$, we construct a continuous-time process $(X^\varepsilon(t), \alpha^\varepsilon(t))$, $t \geq 0$ such that,

$$X^\varepsilon(0) = X_0, \ \alpha^\varepsilon(0) = \alpha_0, \tag{18}$$

$$X^\varepsilon(t) = X_n, \ \alpha^\varepsilon(t) = \alpha_n, \ \text{ for } n\varepsilon \leq t < (n+1)\varepsilon, \ n = 0, 1, \ldots, N-1, \tag{19}$$

where for given $(X_n, \alpha_n) = (x, i)$, (X_{n+1}, α_{n+1}) is determined as following: First for the discrete Markov chain $\{\alpha_n, n \geq 0\}$, the transition probability matrix is given by

$$P^\varepsilon = I + \varepsilon Q, \tag{20}$$

that implies

$$P\{\alpha_{n+1} = j | \alpha_n = i\} = \delta_{ij} + \varepsilon q_{ij}, \ \ 1 \leq i, j \leq m_0, \ \ 0 \leq n \leq N-1, \tag{21}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. It can be shown by applying Theorem 3.1 in Ref. 27 that the interpolated process $\alpha^\varepsilon(\cdot)$ converges weakly to the continuous Markov process $\alpha(\cdot)$ generated by Q.

Next, we assume that for given $(X_n, \alpha_n) = (x, i)$, X_{n+1} can take n_0 different values $X^{\varepsilon,l}(x, i)$, $l = 1, \ldots, n_0$ with conditional probabilities,

$$P\left\{ X_{n+1} = X^{\varepsilon,l}(x,i) \middle| X_n = x, \alpha_n = i \right\} = p^{\varepsilon,l}(x,i), \ l = 1, 2, \ldots n_0, \tag{22}$$

where functions $X^{\varepsilon,l}(\cdot,\cdot) : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ satisfy $-\infty < X^{\varepsilon,l}(x,i) < \infty$ for all $x \in \mathbb{R}$ and each $i \in \mathcal{M}$, and $p^{\varepsilon,l}(\cdot,\cdot) : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ satisfy

$$0 \leq p^{\varepsilon,l}(x,i) \leq 1, \ \ \sum_{l=1}^{n_0} p^{\varepsilon,l}(x,i) = 1, \ \text{ for all } x \in \mathbb{R}, \ i \in \mathcal{M}. \tag{23}$$

Moreover, we assume throughout this section that, for $i \in \mathcal{M}$, $x \in \mathbb{R}$,

$$E\left[X_{n+1} \middle| X_n = x, \alpha_n = i \right] = \sum_{l=1}^{n_0} p^{\varepsilon,l}(x,i) X^{\varepsilon,l}(x,i) = x + \mu(x,i)\varepsilon, \tag{24}$$

and

$$\begin{aligned} E\left[(X_{n+1} - x)^2 \middle| X_n = x, \alpha_n = i \right] &= \sum_{l=1}^{n_0} p^{\varepsilon,l}(x,i) \left(X^{\varepsilon,l}(x,i) - x \right)^2 \\ &= \sigma^2(x,i)\varepsilon + \mu^2(x,i)\varepsilon^2. \end{aligned} \tag{25}$$

Note that these conditions are all satisfied by the regime-switching recombining tree design presented in Section 2 with $n_0 = 3$.

The following lemma presents an upper bound of the second moments of X_n, $0 \le n \le \lfloor \frac{T}{\varepsilon} \rfloor$ which will be used in the weak convergence proof.

Lemma 1.. *Under Assumptions (14), (15), (24) and (25), for any fixed $T > 0$ and any $\varepsilon > 0$, we have*

$$\sup_{0 \le n \le \lfloor T/\varepsilon \rfloor} E\left[X_n\right]^2 \le (X_0^2 + KT)e^{KT} < \infty. \tag{26}$$

Proof. Using (24) and (25) we have,

$$\begin{aligned} E\left[X_{n+1}^2 \middle| X_n, \alpha_n\right] &= E\left[\left(X_n + (X_{n+1} - X_n)\right)^2 \middle| X_n, \alpha_n\right] \\ &= X_n^2 + 2X_n \mu\left(X_n, \alpha_n\right)\varepsilon + \sigma^2\left(X_n, \alpha_n\right)\varepsilon + \mu^2\left(X_n, \alpha_n\right)\varepsilon^2. \end{aligned}$$

Using (15) we have,

$$\mu^2\left(X_n, \alpha_n\right) \le K\left(1 + X_n^2\right) \text{ and } \sigma^2\left(X_n, \alpha_n\right) \le K\left(1 + X_n^2\right).$$

It follows that

$$E\left[X_{n+1}^2 \middle| X_n, \alpha_n\right] \le X_n^2 + K\varepsilon\left(1 + X_n^2\right). \tag{27}$$

Taking the expectation on both sides of (27) leads to

$$E\left[X_{n+1}^2\right] \le E\left[X_n^2\right] + K\varepsilon + K\varepsilon E\left[X_n^2\right]. \tag{28}$$

Iterating (28), we obtain

$$E\left[X_n^2\right] \le X_0^2 + Kn\varepsilon + K\varepsilon \sum_{j=0}^{n-1} E\left[X_j^2\right]. \tag{29}$$

Applying the Gronwall's inequality yields

$$E\left[X_n^2\right] \le (X_0^2 + Kn\varepsilon)e^{KT} \le (X_0^2 + KT)e^{KT},$$

and then (26) follows. □

It follows immediately that

$$\sup_{0 \le t \le T} E\left[X^\varepsilon(t)\right]^2 \le K < \infty. \tag{30}$$

Next, we establish the tightness of the continuous processes $\{X^\varepsilon(t) : 0 \le t \le T\}$ in a suitable function space. To this end, we rewrite the approximation (22) into the following iterative equation:

$$X_{n+1} = X_n + \mu\left(X_n, \alpha_n\right)\varepsilon + \xi\left(X_n, \alpha_n\right), \tag{31}$$

where the random functions $\xi(\cdot,\cdot)$ are defined as following: For each $x \in \mathbb{R}$ and each $i \in \mathcal{M}$, $\xi(x,i)$ takes values from the set of n_0 values $\{X^{\varepsilon,l}(x,i) - x - \mu(x,i)\varepsilon, l = 1,\ldots,n_0\}$ with probabilities $p^{\varepsilon,l}(x,i)$, $l = 1,2,\ldots n_0$, as defined in (22) and (23). It follows from (24) and (25) that

$$\begin{aligned} &E\left[\xi\left(X_n,\alpha_n\right)\Big|X_n = x, \alpha_n = i\right] = 0, \\ &E\left[\xi^2\left(X_n,\alpha_n\right)\Big|X_n = x, \alpha_n = i\right] = \sigma^2(x,i)\varepsilon. \end{aligned} \tag{32}$$

Lemma 2.. *Under Assumptions (14), (15), (24) and (25), the interpolation process $\{X^\varepsilon(t) : 0 \le t \le T\}$ is tight in $D[0,T]$, the space of functions that are right continuous and have left limits, endowed with the Skorohod topology.*

Proof. For any $\delta > 0$, $t > 0$, $0 \le s \le \delta$ and $t + s \le T$, we have,

$$\begin{aligned} E\left[X^\varepsilon(t+s) - X^\varepsilon(t)\right]^2 &= E\left[\sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}(X_{k+1} - X_k)\right]^2 \\ &= E\left[\sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\left(\mu\left(X_k,\alpha_k\right)\varepsilon + \xi\left(X_k,\alpha_k\right)\right)\right]^2 \\ &\le 2\varepsilon^2 E\left[\sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\mu\left(X_k,\alpha_k\right)\right]^2 + 2E\left[\sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\xi\left(X_k,\alpha_k\right)\right]^2. \end{aligned}$$

Note that for simplicity, in the above we have used t/ε and $(t+s)/\varepsilon$ for the integer parts of t/ε and $(t+s)/\varepsilon$, respectively. We will adapt this convention for the rest of the paper.

Note that

$$\begin{aligned} \varepsilon^2 E\left[\sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\mu\left(X_k,\alpha_k\right)\right]^2 &\le \varepsilon\delta \sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1} E\left[\mu^2\left(X_k,\alpha_k\right)\right] \\ &\le \varepsilon\delta K \sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\left(1 + E[X_k^2]\right) \le K\delta^2, \end{aligned}$$

and

$$\begin{aligned}
&E\left[\sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\xi\left(X_k,\alpha_k\right)\right]^2 \\
&= \sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1} E\left[\xi^2\left(X_k,\alpha_k\right)\right] + 2\sum_{k<l} E\left[\xi\left(X_k,\alpha_k\right)\xi\left(X_l,\alpha_l\right)\right] \\
&= \sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1} E\left[E_k\left[\xi^2\left(X_k,\alpha_k\right)\right]\right] + 2\sum_{k<l} E\left[\xi\left(X_k,\alpha_k\right)E_l\left[\xi\left(X_l,\alpha_l\right)\right]\right] \\
&= \varepsilon \sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1} E\left[\sigma^2\left(X_k,\alpha_k\right)\right] \le K\varepsilon \sum_{k=t/\varepsilon}^{(t+s)/\varepsilon-1}\left(1+E[X_k^2]\right) \le K\delta,
\end{aligned}$$

where we have used E_k for the conditional expectation given (X_k, α_k). We have

$$E\left[X^\varepsilon(t+s) - X^\varepsilon(t)\right]^2 \le K(\delta + \delta^2).$$

Then it follows that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} E\left[X^\varepsilon(t+s) - X^\varepsilon(t)\right]^2 = 0.$$

Then the tightness criterion (see [18, P47]) implies that $\{X^\varepsilon(\cdot)\}$ is tight in $D[0,T]$. □

Note that it can be shown by applying Theorem 3.1 in Ref. 27 that the interpolated process $\{\alpha^\varepsilon(\cdot)\}$ is tight. It then follows that $\{X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ is tight in $D([0,T];\mathbb{R}\times\mathcal{M})$, the space of functions that are defined on $D[0,T]$ taking values in $\mathbb{R}\times\mathcal{M}$, and that are right continuous and have left limits, endowed with the Skorohod topology.

Since $\{X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ is tight, by the Prohorov's Theorem, it has convergent subsequences. We select such a subsequence and still denote it by $\{X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ for notational simplicity. Denote the limit by $\{X(\cdot), \alpha(\cdot)\}$. By Skorohod representation, we may assume without loss of generality that the subsequence $\{X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ converges to $\{X(\cdot), \alpha(\cdot)\}$ w.p.1, and that the convergence is uniform on any bounded interval. Next we apply the Martingale problem formulation method (Refs. 25,28) to characterize the limit process $\{X(\cdot), \alpha(\cdot)\}$.

Theorem 1.. *Under Assumptions (14), (15), (24) and (25), the interpolation process $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ converges weakly to $(X(\cdot), \alpha(\cdot))$ as $\varepsilon \to 0$ such that $(X(\cdot), \alpha(\cdot))$ is the solution of the martingale problem associated with operator $\mathcal{L}$ defined by (16).*

Proof. We are to show that for each $i \in \mathcal{M}$ and for any function $\varphi(\cdot, i) \in C_0^2$,

$$M_\varphi(t) := \varphi(X(t), \alpha(t)) - \varphi(X_0, \alpha_0) - \int_0^t \mathcal{L}\varphi(X(s), \alpha(s))\, ds \qquad (33)$$

is a martingale. This is done via a series of approximations that average out the unwanted terms.

Following Refs. 25,28, the claim (33) will be verified if we can show that for each $i \in \mathcal{M}$, any bounded and continuous function $h_j(\cdot, i)$, any positive integer κ, any $0 < t_j \le t$ with $j \le \kappa$, $s > 0$, and $t + s \le T$,

$$E\left\{ \prod_{j=1}^{\kappa} h_j(X(t_j), \alpha(t_j)) \Big[\varphi(X(t+s), \alpha(t+s)) - \varphi(X(t), \alpha(t)) - \int_t^{t+s} \mathcal{L}\varphi(X(u), \alpha(u))\, du \Big] \right\} = 0. \qquad (34)$$

In what follows we show that (34) holds for the pair $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ and hence for the pair $(X(\cdot), \alpha(\cdot))$. The proof is divided into four steps.

Step 1. We begin by choosing a sequence of positive integers $\{m_\varepsilon\}$ satisfying $m_\varepsilon \to \infty$ as $\varepsilon \to 0$ but $\Delta_\varepsilon := \varepsilon m_\varepsilon \to 0$. Again, we use t/Δ_ε and $(t+s)/\Delta_\varepsilon$ for their integer parts (to avoid the use of the floor function notation). Introduce the index set

$$\chi_l^\varepsilon = \{k : l m_\varepsilon \le k \le (l+1)m_\varepsilon - 1\}. \qquad (35)$$

Then we can rewrite $\varphi(X^\varepsilon(t+s), \alpha^\varepsilon(t+s)) - \varphi(X^\varepsilon(t), \alpha^\varepsilon(t))$ as

$$\begin{aligned} &\varphi(X^\varepsilon(t+s), \alpha^\varepsilon(t+s)) - \varphi(X^\varepsilon(t), \alpha^\varepsilon(t)) \\ &= \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \left[\varphi(X_{(l+1)m_\varepsilon}, \alpha_{(l+1)m_\varepsilon}) - \varphi(X_{(l+1)m_\varepsilon}, \alpha_{l m_\varepsilon}) \right] \\ &\quad + \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \left[\varphi(X_{(l+1)m_\varepsilon}, \alpha_{l m_\varepsilon}) - \varphi(X_{l m_\varepsilon}, \alpha_{l m_\varepsilon}) \right]. \end{aligned} \qquad (36)$$

Step 2. We work on the second summation (the third line) of (36) in this step. Noting that

$$\begin{aligned} &\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \left[\varphi(X_{(l+1)m_\varepsilon}, \alpha_{l m_\varepsilon}) - \varphi(X_{l m_\varepsilon}, \alpha_{l m_\varepsilon}) \right] \\ &\qquad = \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \sum_{k: k \in \chi_l^\varepsilon} \left[\varphi(X_{k+1}, \alpha_{l m_\varepsilon}) - \varphi(X_k, \alpha_{l m_\varepsilon}) \right], \end{aligned}$$

we have,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} E\left\{\prod_{j=1}^{\kappa} h_j(X^{\varepsilon}(t_j),\alpha^{\varepsilon}(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\left[\varphi(X_{(l+1)m_\varepsilon},\alpha_{lm_\varepsilon})-\varphi(X_{lm_\varepsilon},\alpha_{lm_\varepsilon})\right]\right\}\\
&=\lim_{\varepsilon\to 0} E\left\{\prod_{j=1}^{\kappa} h_j(X^{\varepsilon}(t_j),\alpha^{\varepsilon}(t_j))\left[\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\varphi'_x(X_k,\alpha_{lm_\varepsilon})\mu(X_k,\alpha_k)\varepsilon\right.\right.\\
&+\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\varphi'_x(X_k,\alpha_{lm_\varepsilon})\xi(X_k,\alpha_k)\\
&+\frac{1}{2}\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\mu^2(X_k,\alpha_k)\varepsilon^2\\
&+\frac{1}{2}\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\xi^2(X_k,\alpha_k)\\
&+\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\mu(X_k,\alpha_k)\xi(X_k,\alpha_k)\varepsilon\\
&\left.\left.+\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\int_0^1\left[\varphi''_{xx}(X_k+u\widetilde{\Delta x}_k,\alpha_{lm_\varepsilon})-\varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\right]\widetilde{\Delta x}_k^{\,2}\,du\right]\right\},
\end{aligned}\tag{37}$$

where $\widetilde{\Delta x}_k = \mu(X_k,\alpha_k)\varepsilon + \xi(X_k,\alpha_k)$. Since $\varphi''_{xx}(\cdot,i)$ is continuous with compact support, the weak convergence of $(X^{\varepsilon}(\cdot),\alpha^{\varepsilon}(\cdot))$, the Skorohod representation, and the boundedness of $h_j(\cdot,i)$ then yield

$$\begin{aligned}
\lim_{\varepsilon\to 0} E\Bigg\{\prod_{j=1}^{\kappa} h_j(X^{\varepsilon}(t_j),\alpha^{\varepsilon}(t_j))\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1}\sum_{k:k\in\chi_l^\varepsilon}\int_0^1\Big[\varphi''_{xx}(X_k+u\widetilde{\Delta x}_k,\alpha_{lm_\varepsilon})\\
-\varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\Big]\widetilde{\Delta x}_k^{\,2}\,du\Bigg\}=0.
\end{aligned}\tag{38}$$

Since $\varphi_x'(\cdot, i)$ and $\mu(\cdot, i)$ are continuous for each $i \in \mathcal{M}$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} E\left\{ \prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j), \alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \sum_{k:k\in\chi_l^\varepsilon} \varphi_x'(X_k, \alpha_{lm_\varepsilon})\mu(X_k, \alpha_k)\varepsilon \right\} \\
&= \lim_{\varepsilon\to 0} E\left\{ \prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j), \alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \varepsilon \sum_{k:k\in\chi_l^\varepsilon} \varphi_x'(X_{lm_\varepsilon}, \alpha_{lm_\varepsilon})\mu(X_{lm_\varepsilon}, \alpha_k) \right\} \\
&= \lim_{\varepsilon\to 0} E\left\{ \prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j), \alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \varepsilon \sum_{k:k\in\chi_l^\varepsilon} E_{lm_\varepsilon} \right. \\
&\qquad\qquad \left. \times \left[\sum_{i=1}^{m_0} \varphi_x'(X_{lm_\varepsilon}, \alpha_{lm_\varepsilon})\mu(X_{lm_\varepsilon}, i) I_{\{\alpha_k = i\}} \right] \right\}.
\end{aligned}$$

Note that when $\varepsilon \to 0$ and $l\varepsilon m_\varepsilon = l\Delta_\varepsilon \to u$, $(l+1)\Delta_\varepsilon \to u$ and for any k satisfying $lm_\varepsilon \le k < (l+1)m_\varepsilon$, $\varepsilon k \to u$. In addition, in view of the weak convergence of $\alpha^\varepsilon(\cdot)$ to $\alpha(\cdot)$ and the Skorohod representation (without changing notation), we may assume that $I_{\{\alpha^\varepsilon(u)=i\}} \to I_{\{\alpha(u)=i\}}$ w.p.1. As a result, for the above limit, we have,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} E\left\{ \prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j), \alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \varepsilon \sum_{k:k\in\chi_l^\varepsilon} E_{lm_\varepsilon} \right. \\
&\qquad\qquad \left. \times \left[\sum_{i=1}^{m_0} \varphi_x'(X_{lm_\varepsilon}, \alpha_{lm_\varepsilon})\mu(X_{lm_\varepsilon}, i) I_{\{\alpha_k = i\}} \right] \right\} \\
&= \lim_{\varepsilon\to 0} E\left\{ \prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j), \alpha^\varepsilon(t_j)) \sum_{i=1}^{m_0} \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon - 1} \Delta_\varepsilon \varphi_x'(X^\varepsilon(l\Delta_\varepsilon), \alpha^\varepsilon(l\Delta_\varepsilon))\mu(X^\varepsilon(l\Delta_\varepsilon), i) \right. \\
&\qquad\qquad \left. \times \frac{1}{m_\varepsilon} E_{lm_\varepsilon} \left[\sum_{k:k\in\chi_l^\varepsilon} I_{\{\alpha^\varepsilon(\varepsilon k)=i\}} \right] \right\} \\
&= E\left\{ \prod_{j=1}^{\kappa} h_j(X(t_j), \alpha(t_j)) \sum_{i=1}^{m_0} \int_t^{t+s} \varphi_x'(X(u), \alpha(u))\mu(X(u), i) I_{\{\alpha(u)=i\}}\, du \right\} \\
&= E\left\{ \prod_{j=1}^{\kappa} h_j(X(t_j), \alpha(t_j)) \int_t^{t+s} \varphi_x'(X(u), \alpha(u))\mu(X(u), \alpha(u))\, du \right\}.
\end{aligned} \tag{39}$$

In view of (32), we have

$$
\begin{aligned}
&E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi'_x(X_k,\alpha_{lm_\varepsilon})\xi(X_k,\alpha_k)\right\}\\
&= E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi'_x(X_k,\alpha_{lm_\varepsilon})E_k\left[\xi(X_k,\alpha_k)\right]\right\}\\
&= 0,
\end{aligned}
\tag{40}
$$

and similarly,

$$
\begin{aligned}
E\Bigg\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})&\\
\times \mu(X_k,\alpha_k)\xi(X_k,\alpha_k)\varepsilon\Bigg\} = 0.&
\end{aligned}
\tag{41}
$$

Using the boundedness of $\varphi''_{xx}(\cdot,i)$, $h_j(\cdot,i)$, the linear growth condition (15) for $\mu(\cdot,i)$, and the estimate (26), we have,

$$
\begin{aligned}
&E\left|\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\mu^2(X_k,\alpha_k)\varepsilon^2\right|\\
&\le K \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varepsilon^2 E\left[\mu^2(X_k,\alpha_k)\right] \le K\varepsilon^2 \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \left(1+E\left[X_k^2\right]\right)\\
&\le K\varepsilon^2\cdot\frac{T}{\varepsilon} = O(\varepsilon) \to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
\tag{42}
$$

In view of (32), we have

$$
\begin{aligned}
&E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\xi^2(X_k,\alpha_k)\right\}\\
&= E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})E_k\left[\xi^2(X_k,\alpha_k)\right]\right\}\\
&= E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\sigma^2(X_k,\alpha_k)\varepsilon\right\}.
\end{aligned}
$$

By using a similar argument to the one leading to (39), we can show that

$$\begin{aligned}&\lim_{\varepsilon\to 0} E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \varphi''_{xx}(X_k,\alpha_{lm_\varepsilon})\xi^2(X_k,\alpha_k)\right\}\\ &= E\left\{\prod_{j=1}^{\kappa} h_j(X(t_j),\alpha(t_j)) \int_t^{t+s} \varphi''_{xx}(X(u),\alpha(u))\sigma^2(X(u),\alpha(u))\,du\right\}.\end{aligned} \tag{43}$$

Using (38)-(43) in (37), we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0} E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \left[\varphi(X_{(l+1)m_\varepsilon},\alpha_{lm_\varepsilon}) - \varphi(X_{lm_\varepsilon},\alpha_{lm_\varepsilon})\right]\right\}\\ &\quad = E\Bigg\{\prod_{j=1}^{\kappa} h_j(X(t_j),\alpha(t_j)) \int_t^{t+s} \Big[\varphi'_x(X(u),\alpha(u))\mu(X(u),\alpha(u))\\ &\qquad + \varphi''_{xx}(X(u),\alpha(u))\sigma^2(X(u),\alpha(u))\Big]\,du\Bigg\}.\end{aligned} \tag{44}$$

Step 3. Now we work on the first summation (the second line) of (36). Using the continuity of $\varphi(\cdot,i)$ for each $i\in\mathcal{M}$, we can argue along the same line as in Step 2 that the limit of

$$\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \left[\varphi(X_{(l+1)m_\varepsilon},\alpha_{(l+1)m_\varepsilon}) - \varphi(X_{(l+1)m_\varepsilon},\alpha_{lm_\varepsilon})\right]$$

is the same as that of

$$\sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \left[\varphi(X_{lm_\varepsilon},\alpha_{(l+1)m_\varepsilon}) - \varphi(X_{lm_\varepsilon},\alpha_{lm_\varepsilon})\right] + \eta_\varepsilon$$

where $\eta_\varepsilon \to 0$ in probability uniformly in t, as $\varepsilon \to 0$. We then have,

$$\begin{aligned}
&E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \left[\varphi(X_{lm_\varepsilon},\alpha_{(l+1)m_\varepsilon}) - \varphi(X_{lm_\varepsilon},\alpha_{lm_\varepsilon})\right]\right\} \\
&= E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \left[\varphi(X_{lm_\varepsilon},\alpha_{k+1}) - \varphi(X_{lm_\varepsilon},\alpha_k)\right]\right\} \\
&= E\Bigg\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \sum_{i=1}^{m_0} E_k\{[\varphi(X_{lm_\varepsilon},\alpha_{k+1}) \\
&\qquad - \varphi(X_{lm_\varepsilon},\alpha_i)]I_{\{\alpha_k=i\}}\}\Bigg\} \\
&= E\Bigg\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \sum_{i=1}^{m_0} \Bigg[\sum_{j=1}^{m_0} \varphi(X_{lm_\varepsilon},j) \\
&\qquad \times P\{\alpha_{k+1}=j|\alpha_k=i\} - \varphi(X_{lm_\varepsilon},i)\Bigg] I_{\{\alpha_k=i\}}\Bigg\} \\
&= E\Bigg\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \sum_{i=1}^{m_0} \Bigg[\sum_{j=1}^{m_0} \varphi(X_{lm_\varepsilon},j)(\delta_{ij}+\varepsilon q_{ij}) \\
&\qquad - \varphi(X_{lm_\varepsilon},i)\Bigg] I_{\{\alpha_k=i\}}\Bigg\} \\
&= E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \sum_{l=t/\Delta_\varepsilon}^{(t+s)/\Delta_\varepsilon-1} \sum_{k:k\in\chi_l^\varepsilon} \sum_{i=1}^{m_0}\sum_{j=1}^{m_0} \varepsilon q_{ij}\varphi(X_{lm_\varepsilon},j) I_{\{\alpha_k=i\}}\right\} \\
&\to E\left\{\prod_{j=1}^{\kappa} h_j(X(t_j),\alpha(t_j)) \int_t^{t+s} Q\varphi\left(X(u),\cdot\right)\left(\alpha(u)\right) du \quad \text{as } \varepsilon\to 0,\right.
\end{aligned} \tag{45}$$

in which (21) is used for the one-step transition probability of $\{\alpha_n, n \geq 0\}$.

Step 4. By combining Step 1 to 3, we have

$$\begin{aligned}
&E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j),\alpha^\varepsilon(t_j)) \left[\varphi(X^\varepsilon(t+s),\alpha^\varepsilon(t+s)) - \varphi(X^\varepsilon(t),\alpha^\varepsilon(t))\right]\right\} \\
&\to E\left\{\prod_{j=1}^{\kappa} h_j(X(t_j),\alpha(t_j)) \int_t^{t+s} \mathcal{L}\varphi(X(u),\alpha(u))\, du\right\} \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

On the other hand, by virtue of the weak convergence of $(X^\varepsilon(\cdot),\alpha^\varepsilon(\cdot))$, the

Skorohod representation, and the continuity of $h_j(\cdot, i)$ and $\varphi(\cdot, i)$, we have,

$$E\left\{\prod_{j=1}^{\kappa} h_j(X^\varepsilon(t_j), \alpha^\varepsilon(t_j))\left[\varphi(X^\varepsilon(t+s), \alpha^\varepsilon(t+s)) - \varphi(X^\varepsilon(t), \alpha^\varepsilon(t))\right]\right\}$$
$$\to E\left\{\prod_{j=1}^{\kappa} h_j(X(t_j), \alpha(t_j))\left[\varphi(X(t+s), \alpha(t+s)) - \varphi(X(t), \alpha(t))\right]\right\} \quad \text{as } \varepsilon \to 0.$$

Hence equation (34) is verified, and the desired result follows and the proof of Theorem 1 is completed. □

4. Concluding Remarks

In this paper we present a multinomial tree method for option pricing when the underlying asset price follows a regime-switching model. The new tree grows linearly as the number of time steps increases. We prove the weak convergence of the discrete tree approximations to the continuous regime-switching process by applying the method of martingale problem formulation.

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