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A FINITE-HORIZON OPTIMAL INVESTMENT AND CONSUMPTION PROBLEM USING REGIME-SWITCHING MODELS

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This paper is concerned with a finite-horizon optimal investment and consumption problem in continuous-time regime-switching models. The market consists of one bond and $n \geq 1$ correlated stocks. An investor distributes his/her wealth among these assets and consumes at a non-negative rate. The market parameters (the interest rate, the appreciation rates and the volatilities of the stocks) and the utility functions are assumed to depend on a continuous-time Markov chain with a finite number of states. The objective is to maximize the expected discounted total utility of consumption and the expected discounted utility from terminal wealth. We solve the optimization problem by applying the stochastic control methods to regime-switching models. Under suitable conditions, we prove a verification theorem. We then apply the verification theorem to a power utility function and obtain, up to the solution of a system of coupled ordinary differential equations, an explicit solution of the value function and the optimal investment and consumption policies. We illustrate the impact of regime-switching on the optimal investment and consumption policies with numerical results and compare the results with the classical Merton problem that has only a single regime.

Keywords: Optimal investment and consumption; regime-switching model; Markov chain; stochastic control; Hamilton–Jacobi–Bellman (HJB) equation.

1. Introduction

The study of Merton's optimal investment and consumption problems (Merton 1969, 1971) using Markov regime-switching models has gained increasing attention in recent years in the financial mathematics community. Aiming to include the response to macroeconomic conditions of the individual asset price behavior, the regime-switching models are capable of capturing the dynamical change of the asset prices across different stages of business cycles. They are particularly suitable for problems involving long time horizon, as in the case of investment and consumption problems.

Zariphopoulou (1992) studied an infinite time horizon problem of maximizing the expected discounted total utility from consumption in a market with one bond and one stock. In her model, only the rate of return of the stock is assumed to depend on a Markov chain and a diffusion term (driven by a Brownian motion) is not included in the stock price model. Bäuerle & Rieder (2004) solved the problem of maximizing the expected utility from terminal wealth for a market with one bond and one stock. For the stock price dynamics, they used a geometric Brownian motion (GBM) whose coefficients (the interest rate, the appreciation rate and the volatility) depend on an external Markov chain, assumed observable at any time t. Later on, they treated the same problem for an unobservable Markov chain (Bäuerle & Rieder 2007 and Rieder & Bäuerle 2005). To use the Wonham filter for estimating the probability distribution of the Markov chain, they assumed a constant volatility in both papers (Bäuerle & Rieder 2007 and Rieder & Bäuerle 2005), and used a regime-dependent appreciation rate in Rieder & Bäuerle (2005), and a regime-dependent jump density in a jump diffusion model in Bäuerle & Rieder (2007) for the stock prices. Honda (2003) also considered the case of unobservable Markov chain and used the Wonham filter. The optimization criterion includes the expected discounted total utility from consumption and the expected utility of the terminal wealth. Honda used a constant volatility for stock price and assumed the Markov chain has two states (two-regime case). Sass & Haussmann (2004) studied a problem of maximizing the expected utility of terminal wealth in a market with one bond and multiple stocks. The drifts of the stocks are governed by an unobservable Markov chain whose estimation was done by HMM filtering.

In addition to providing more accurate representations for financial asset dynamics, regime-switching models preserve more or less the analytical tractability of the classical Merton problem and allow people to derive analytical results for certain problems. Sotomayor & Cadenillas (2009) treated an infinite time horizon problem in a regime-switching market with one bond and multiple stocks. The model coefficients are all dependent on a Markov chain which is assumed observable. Moreover, they also allow the utility functions to depend on the regime, which extends the previous works dealing with regime-switching. Applying stochastic control approaches, Sotomayor & Cadenillas (2009) obtained explicit optimal investment and consumption policies for four specific utility functions of the Hyperbolic Absolute Risk Aversion (HARA) type.

In this paper, we present a finite-horizon problem of optimal investment and consumption with regime-dependent market parameters and utility functions. The model setup is very similar to Sotomayor & Cadenillas (2009). However, our objective of optimization consists of both expected discounted total utility from consumption and the expected utility of the terminal wealth if bankruptcy does not happen by a pre-specified maturity time. As a result, the optimal consumption policy also depends on the time as well as the regime, while the optimal investment policy depends only on the regime, not on the time. Under suitable conditions, we prove a verification theorem for the finite-horizon control problem with regime-switching considered in this paper, that is in contrast with the one established for the infinite-horizon problem (Sotomayor & Cadenillas (2009), Theorem 3.1). We then apply the verification theorem to a regime-dependent power utility function and obtain, up to the solution of a system of coupled nonlinear ordinary differential equations (ODEs), an explicit solution of the optimal investment and consumption policies as well as the value function. We study the effect of regime-switching on the optimal investment and consumption policies by numerical experiments for which we compare the results with the classical Merton problem that has only a single regime.

To conclude the introduction, we cite several additional articles from the literature that use regime-switching models. Zhang *et al.* (2010) considered the problem of maximizing the expected utility from terminal wealth in an enlarged market by introducing a set of so-called Markovian jumps securities in order to first complete the market. Zhou & Yin (2003) studied the Markowitz's mean-variance portfolio selection problem in regime-switching models and derived explicit solutions for the efficient frontiers, based on solutions of two systems of linear ODEs. Zhang & Yin (2004) developed nearly optimal policies for optimal asset allocation by using a special hierarchical structure of the underlying Markov chain. Liu (2013) studied an infinite-horizon problem of optimal investment and consumption with proportional transaction costs in continuous-time regime-switching models and extended the analysis and results of Shreve & Soner (1994) for a power utility function to the regime-switching case. More recently, Fei (2013) considered the impact of inflation on the optimal consumption in the Markovian regime-switching model, and solved the problem of maximizing the expected utility of consumption discounted by the inflation. Fu *et al.* (2014) studied the problem of maximizing the expected utility of the terminal wealth of a portfolio that contains a stock, a bond, and an option.

The paper is organized as follows. Section 2 presents the mathematical formulation of the optimal investment and consumption problem in regime-switching models, and the associated Hamilton–Jacobi–Bellman (HJB) equation which is given by a system of coupled second-order nonlinear partial differential equations. Section 3 establishes a verification theorem for the solution of the optimal control problem. Section 4 treats a power utility function and derives explicit solutions (up to the solution of a system of coupled nonlinear ODEs) of the optimal investment and consumption policies. Section 5 reports numerical results. Section 6 provides further remarks and concludes the paper.

2. Problem Formulation and HJB Equation

Given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions of right continuity and completeness. Let $B(t) = (B_1(t), B_2(t), \ldots, B_d(t))^T \in \mathbb{R}^d$ be a d-dimensional standard Brownian motion where the superscript T denotes transpose of vector. Let $\alpha(t)$ be a continuous-time Markov chain valued in a finite set $\mathcal{M} := \{1, \ldots, m_0\}$ with $m_0 > 0$ fixed. Let $Q = (q_{ij})_{m_0 \times m_0}$ be the intensity matrix (or the generator) of $\alpha(t)$. Q satisfies: (I)

$q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$ for each $i = 1, \ldots, m_0$. In this paper, we assume that $\alpha(t)$ is observable and independent of the Brownian motion $B(t)$.

We consider a financial market with one bond and $n \geq 1$ stocks. The bond price $S_0(t)$ follows the equation

$$dS_0(t) = r(\alpha(t)) S_0(t) dt, \tag{2.1}$$

where $r(\alpha(t))$ is the interest rate depending on $\alpha(t)$. For $1 \leq k \leq n$, the kth stock price $S_k(t)$ follows the stochastic differential equation (SDE)

$$dS_k(t) = S_k(t) \left[\mu_k(\alpha(t)) dt + \sum_{j=1}^{d} \sigma_{kj}(\alpha(t)) dB_j(t) \right], \tag{2.2}$$

where $\mu_k(\alpha(t))$ and $\sigma_{kj}(\alpha(t))$, $j = 1, \ldots, d$ are the appreciation rate and the volatility coefficients of the kth stock.

An investor distributes his/her wealth among the bond and the stocks, and consumes at a non-negative rate. Let $\pi_k(t)$ denote the fraction of wealth invested in the kth stock at time t. Then $1 - \sum_{k=1}^{n} \pi_k(t)$ is the fraction of wealth invested in the bond at time t. Let $c(t)$ denote the consumption rate at time t. Let $X(t)$ be the investor's wealth at time t. Then using (2.1) and (2.2) we obtain the stochastic differential equation for the wealth process $\{X(t), t \geq 0\}$:

$$\begin{aligned} dX(t) = X(t) r(\alpha(t)) dt &+ \sum_{k=1}^{n} X(t) \pi_k(t) [\mu_k(\alpha(t)) - r(\alpha(t))] dt - c(t) dt \\ &+ \sum_{k=1}^{n} X(t) \pi_k(t) \sum_{j=1}^{d} \sigma_{kj}(\alpha(t)) dB_j(t). \end{aligned} \tag{2.3}$$

Let $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))^T \in \mathbb{R}^n$. For $i \in \mathcal{M}$, let $\mu(i) = (\mu_1(i), \ldots, \mu_n(i))^T \in \mathbb{R}^n$, $\sigma(i) = (\sigma_{kj}(i))_{n \times d} \in \mathbb{R}^{n \times d}$ be respectively the vector of appreciation rates and the volatility matrix in regime i. Moreover, let $\mathbb{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ denote the unit vector. Then we can rewrite the wealth process (2.3) as:

$$\begin{aligned} dX(t) = X(t) r(\alpha(t)) dt &+ X(t) \pi^T(t) [\mu(\alpha(t)) - r(\alpha(t)) \mathbb{1}] dt - c(t) dt \\ &+ X(t) \pi^T(t) \sigma(\alpha(t)) dB(t). \end{aligned} \tag{2.4}$$

Now we introduce the notion of admissible control. Fix $0 \leq t < T < \infty$, $X(t) = x > 0$ and $\alpha(t) = i \in \mathcal{M}$. In what follows, we use E^{txi} to denote the conditional expectation given t, $x(t) = x$ and $\alpha(t) = i$.

Definition 2.1. A stochastic process $u(\cdot) = \{u(s), t \leq s \leq T\}$ where $u(s) = (\pi^T(s), c(s))^T \in \mathbb{R}^{n+1}$, is called an admissible control process if the following conditions hold:

(1) $u(\cdot)$ is $\mathbb{F}$-adapted,

(2)

$$E^{txi}\left[\int_t^T |\pi(s)|^2\, ds\right] < \infty, \tag{2.5}$$

(3)

$$c(s) \geq 0, \text{a.s.}, \quad t \leq s \leq T, \quad \text{and} \quad E^{txi}\left[\int_t^T c(s)\, ds\right] < \infty, \tag{2.6}$$

(4)

$$E^{txi}\left[\sup_{t\leq s\leq T} |X(s)|^2\right] < \infty. \tag{2.7}$$

Let $\mathcal{A}_{txi}$ denote the collection of admissible controls with respect to the initial time t and state $(x(t), \alpha(t)) = (x, i)$.

Given an admissible control $u(\cdot) \in \mathcal{A}_{txi}$, let X^u denote the solution to the wealth equation (2.4) when $u(\cdot)$ is being used. Let

$$\tau^u = \inf\{s \geq t : X^u(s) \leq 0\} \tag{2.8}$$

be the stopping time for bankruptcy.

Sotomayor & Cadenillas (2009) provide a discussion on the necessity of using regime-dependent utility functions in order to better reflect the investor's risk preferences under different market regimes. In this paper, we consider a class of utility functions that depend on the regime α. Let $U : [0, \infty) \times \mathcal{M} \to \mathbb{R}$ be the utility function with the following properties: for each $i \in \mathcal{M}$, $U(0, i) = 0$, $U'(x, i) > 0$, $U''(x, i) < 0$ for $x > 0$, $\lim_{x\to 0^+} U'(x, i) = \infty$, $\lim_{x\to\infty} U'(x, i) = 0$, where U' and U'' denote the first- and second-order derivatives of U with respect to x.

Given t, $X(t) = x$, $\alpha(t) = i$ and an admissible control $u(\cdot) \in \mathcal{A}_{txi}$, define the objective functional by

$$J(t, x, i; u(\cdot)) = E^{txi}\left[\int_t^{\tau^u\wedge T} \mathrm{e}^{-\beta(s-t)} U(c(s), \alpha(s))ds \right.$$
$$\left. + I_{\{\tau^u > T\}}\mathrm{e}^{-\beta(T-t)} U(X^u(T), \alpha(T))\right], \tag{2.9}$$

where $\beta > 0$ is a discount factor. Note that (2.9) includes both expected discounted total utility from consumption and, if bankruptcy does not occur by the maturity time T, the expected utility of the terminal wealth. The objective of the optimization problem is to find an admissible control process $u(\cdot) \in \mathcal{A}_{txi}$ that maximizes the objective functional J. The value function is defined as

$$V(t, x, i) = \sup_{u(\cdot)\in\mathcal{A}_{txi}} J(t, x, i; u(\cdot)) \tag{2.10}$$

with boundary conditions $V(T, x, i) = U(x, i)$, $x > 0$, $i \in \mathcal{M}$ and $V(t, 0, i) = 0$, $0 \leq t \leq T$, $i \in \mathcal{M}$.

We note that different utility functions (e.g. U_1 for consumption and U_2 for the terminal wealth) could be used in the objective functional (2.9). The proof of the verification theorem presented in the next section would be the same except for notation changes for the utility functions.

The standard approach in stochastic optimal control (see, for example, Fleming & Soner 2006) can be followed to drive the associated HJB equation which is a system of m_0 coupled second-order nonlinear partial differential equations given by:

$$\begin{aligned} v_t(t,x,i) &+ \sup_{\pi\in\mathbb{R}^n}\left\{\pi^T[\mu(i)-r(i)\mathbb{1}]xv_x(t,x,i)+\frac{1}{2}\pi^T\Sigma(i)\pi x^2 v_{xx}(t,x,i)\right\} \\ &+ \sup_{c\geq 0}\{U(c,i)-cv_x(t,x,i)\}+r(i)xv_x(t,x,i)-\beta v(t,x,i) \\ &+ \sum_{j\neq i} q_{ij}[v(t,x,j)-v(t,x,i)]=0, \end{aligned} \tag{2.11}$$

where for each $i\in\mathcal{M}$, $\Sigma(i)=\sigma(i)\sigma^T(i)\in\mathbb{R}^{n\times n}$, and $\pi=(\pi_1,\ldots,\pi_n)^T\in\mathbb{R}^n$. The boundary conditions are given by

$$\begin{cases} v(T,x,i)=U(x,i), & x>0, \quad i\in\mathcal{M}, \\ v(t,0,i)=0, & 0\leq t\leq T, \quad i\in\mathcal{M}. \end{cases} \tag{2.12}$$

3. Verification Theorem

The stochastic optimal control theory has been applied extensively in the literature to optimal investment and consumption problems (see, for example, Fleming & Soner 2006 and Pham 2009). In this paper, we apply the methods to solve the problem with regime-switching under consideration. We establish a verification theorem in this section.

Consider a function $v:[0,T]\times[0,\infty)\times\mathcal{M}\to\mathbb{R}$ with the property that for each $i\in\mathcal{M}$, $v(\cdot,\cdot,i)\in C^{1,2}((0,T)\times(0,\infty))$. Given $u=(\pi^T,c)^T\in\mathbb{R}^n\times[0,\infty)$, we introduce the following operator $\mathcal{L}^u$:

$$\begin{aligned} \mathcal{L}^u v(t,x,i) = v_t(t,x,i) &+ \pi^T[\mu(i)-r(i)\mathbb{1}]xv_x(t,x,i)+\frac{1}{2}\pi^T\Sigma(i)\pi x^2 v_{xx}(t,x,i) \\ &+ r(i)xv_x(t,x,i)-cv_x(t,x,i)-\beta v(t,x,i) \\ &+ \sum_{j\neq i} q_{ij}[v(t,x,j)-v(t,x,i)]. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let the function $W:[0,T]\times[0,\infty)\times\mathcal{M}\to\mathbb{R}$ satisfy:*

- *For each $i\in\mathcal{M}$, $W(\cdot,\cdot,i)\in C^{1,2}((0,T)\times(0,\infty))\cap C([0,T]\times[0,\infty))$.*
- $$|W(t,x,i)|\leq C(1+|x|^2), \quad \text{for all } i\in\mathcal{M},\ x>0 \quad \text{and} \quad 0\leq t\leq T, \tag{3.2}$$
  where $C>0$ is a constant.
- *W is a solution of the HJB equation* (2.11) *with the boundary conditions* (2.12).

Then we have:

(a) $W(t,x,i) \geq J(t,x,i;u(\cdot))$ *for any* $0 \leq t < T, x > 0$, $i \in \mathcal{M}$ *and any admissible control process* $u(\cdot) \in \mathcal{A}_{txi}$. *That implies* $W(t,x,i) \geq V(t,x,i)$.

(b) *Given* (t,x,i). *If there exits an admissible control process* $u^*(\cdot) = (\pi^{*,T}(\cdot), c^*(\cdot))^T \in \mathcal{A}_{txi}$ *such that*

$$\pi^*(s) \in \operatorname*{argsup}_{\pi\in\mathbb{R}^n} \left\{ \pi^T[\mu(\alpha(s)) - r(\alpha(s))\mathbb{1}]X^*(s)W_x(s, X^*(s), \alpha(s)) \right.$$
$$\left. + \frac{1}{2}\pi^T\Sigma(\alpha(s))\pi[X^*(s)]^2W_{xx}(s, X^*(s), \alpha(s)) \right\}, \tag{3.3}$$
$$c^*(s) \in \operatorname*{argsup}_{c\geq 0}\{U(c, \alpha(s)) - cW_x(s, X^*(s), \alpha(s))\},$$

for Lebesgue $\times\, \mathcal{P}$ *almost all* $(s,\omega) \in [t, \tau^*(\omega)] \times \Omega$, *then* $W(t,x,i) = V(t,x,i) = J(t,x,i;u^*(\cdot))$, *that is,* $u^*(\cdot)$ *is an optimal control process. Here* $X^*(s)$ *is the solution of the wealth equation* (2.4) *corresponding to the control process* $u^*(\cdot)$ *with* $X^*(t) = x$, $\alpha(t) = i$, *and* τ^* *is the stopping time defined by* (2.8) *for* $X^*(\cdot)$.

Remark 3.2. Theorem 3.1 is an extension of the classical result (Fleming & Soner, 2006, Theorem 3.1) to the case with regime-switching when applied to the optimal investment and consumption problem considered in this paper.

Proof. Given an admissible control $u(\cdot) \in \mathcal{A}_{txi}$, let X^u denote the solution of the wealth equation (2.4) when $u(\cdot)$ is being used. Applying the generalized Itô's formula (see, for example, Buffington & Elliott 2002) to the Markovian-modulated process $\mathrm{e}^{-\beta s}W(s, X^u(s), \alpha(s))$, we obtain:

$$\begin{aligned} d(e^{-\beta s}W(s, X^u(s), \alpha(s))) = {} & e^{-\beta s}\mathcal{L}^{u(s)}W(s, X^u(s), \alpha(s))ds \\ & + e^{-\beta s}W_x(s, X^u(s), \alpha(s))X^u(s)\pi^T(s)\sigma(\alpha(s))dB(s) \\ & + dM_s^W, \end{aligned} \tag{3.4}$$

where $\{M_s^W\}$ is a martingale provided that W is bounded.

Since W is a solution of the HJB equation (2.11), we have

$$\mathcal{L}^u W(t,x,i) + U(c,i) \leq 0. \tag{3.5}$$

If we replace t, x, i, u by s, $X^u(s)$, $\alpha(s)$, $u(s)$ in (3.5), then

$$\mathcal{L}^{u(s)}W(s, X^u(s), \alpha(s)) + U(c(s), \alpha(s)) \leq 0, \quad t \leq s \leq \tau^u. \tag{3.6}$$

Choose an integer k sufficiently large so that $\frac{1}{k} < x < k$. Define

$$\tau_k^u = \inf\left\{ s \geq t : X^u(s) \notin \left(\frac{1}{k}, k\right)\right\}. \tag{3.7}$$

Let $\theta_k = \tau_k^u \wedge T$. Integrating (3.4) from t to θ_k, we have:

$$\begin{aligned} &\mathrm{e}^{-\beta\theta_k} W(\theta_k, X^u(\theta_k), \alpha(\theta_k)) \\ &\quad = \mathrm{e}^{-\beta t} W(t,x,i) + \int_t^{\theta_k} \mathrm{e}^{-\beta s} \mathcal{L}^{u(s)} W(s, X^u(s), \alpha(s)) ds \\ &\qquad + \int_t^{\theta_k} \mathrm{e}^{-\beta s} W_x(s, X^u(s), \alpha(s)) X^u(s) \pi^T(s) \sigma(\alpha(s)) dB(s) \\ &\qquad + \int_t^{\theta_k} dM_s^W. \end{aligned} \tag{3.8}$$

Note that for $t \le s \le \theta_k$, $|X^u(s)| \le k$ hence $W_x(s, X^u(s), \alpha(s))$ is bounded. In view of (2.5), we have:

$$\begin{aligned} &E^{txi} \left[\int_t^{\theta_k} \mathrm{e}^{-2\beta s} \left[W_x(s, X^u(s), \alpha(s)) X^u(s) \right]^2 \pi^T(s) \Sigma(\alpha(s)) \pi(s) ds \right] \\ &\quad \le C_1 E^{txi} \left[\int_t^{\theta_k} \pi^T(s) \Sigma(\alpha(s)) \pi(s) ds \right] \\ &\quad \le C_2 E^{txi} \left[\int_t^{T} |\pi(s)|^2 ds \right] < \infty \end{aligned} \tag{3.9}$$

for some constants $C_1, C_2 > 0$. It follows that

$$\int_t^{\vartheta \wedge \theta_k} \mathrm{e}^{-\beta s} W_x(s, X^u(s), \alpha(s)) X^u(s) \pi^T(s) \sigma(\alpha(s)) dB(s), \quad \vartheta \in [t, T]$$

is a martingale and hence we have

$$E^{txi} \left[\int_t^{\theta_k} \mathrm{e}^{-\beta s} W_x(s, X^u(s), \alpha(s)) X^u(s) \pi^T(s) \sigma(\alpha(s)) dB(s) \right] = 0. \tag{3.10}$$

On the other hand, since $W(s, X^u(s), \alpha(s))$ is bounded for $t \le s \le \theta_k$, we have

$$E^{txi} \left[\int_t^{\theta_k} dM_s^W \right] = 0. \tag{3.11}$$

Taking expectation in (3.8) and using (3.6), we have

$$\begin{aligned} &E^{txi}[\mathrm{e}^{-\beta\theta_k} W(\theta_k, X^u(\theta_k), \alpha(\theta_k))] \\ &\quad = \mathrm{e}^{-\beta t} W(t,x,i) + E^{txi} \left[\int_t^{\theta_k} \mathrm{e}^{-\beta s} \mathcal{L}^{u(s)} W(s, X^u(s), \alpha(s)) ds \right] \\ &\quad \le \mathrm{e}^{-\beta t} W(t,x,i) - E^{txi} \left[\int_t^{\theta_k} \mathrm{e}^{-\beta s} U(c(s), \alpha(s)) ds \right]. \end{aligned} \tag{3.12}$$

It then follows that:

$$W(t,x,i) \geq E^{txi}\left[\int_t^{\theta_k} \mathrm{e}^{-\beta(s-t)}U(c(s),\alpha(s))ds + \mathrm{e}^{-\beta(\theta_k-t)}W(\theta_k, X^u(\theta_k),\alpha(\theta_k))\right]. \tag{3.13}$$

Next, letting $k \to \infty$, we have $\tau_k^u \uparrow \tau^u$, a.s. and thus $\theta_k \uparrow \tau^u \wedge T$, a.s. Since $U(\cdot,\cdot) \geq 0$, using the monotone convergence theorem, we obtain:

$$\lim_{k\to\infty} E^{txi}\left[\int_t^{\theta_k} \mathrm{e}^{-\beta(s-t)}U(c(s),\alpha(s))ds\right] = E^{txi}\left[\int_t^{\tau^u\wedge T} \mathrm{e}^{-\beta(s-t)}U(c(s),\alpha(s))ds\right]. \tag{3.14}$$

On the other hand, since $W \in C([0,T]\times[0,\infty))$ and $\alpha(\theta_k) \to \alpha(\tau^u \wedge T)$ w.p.1 as $k \to \infty$ (note that w.p.1 there are finite many jumps along each sample path of $\alpha(s)$, $t \leq s \leq T$), we conclude that $\mathrm{e}^{-\beta(\theta_k-t)}W(\theta_k, X^u(\theta_k),\alpha(\theta_k)) \to \mathrm{e}^{-\beta(\tau^u\wedge T-t)}W(\tau^u \wedge T, X^u(\tau^u\wedge T),\alpha(\tau^u\wedge T))$ w.p.1 as $k\to\infty$. In view of (3.2), we have:

$$|W(\theta_k, X^u(\theta_k),\alpha(\theta_k))| \leq C[1+|X^u(\theta_k)|^2] \leq C\left[1+\sup_{t\leq s\leq T}|X^u(s)|^2\right]. \tag{3.15}$$

Then (2.7), the dominated convergence theorem and the boundary conditions (2.12) yield:

$$\begin{aligned}
&\lim_{k\to\infty} E^{txi}[\mathrm{e}^{-\beta(\theta_k-t)}W(\theta_k, X^u(\theta_k),\alpha(\theta_k))]\\
&\quad = E^{txi}[\mathrm{e}^{-\beta(\tau^u\wedge T-t)}W(\tau^u\wedge T, X^u(\tau^u\wedge T),\alpha(\tau^u\wedge T))]\\
&\quad = E^{txi}[I_{\{\tau^u>T\}}\mathrm{e}^{-\beta(T-t)}W(T, X^u(T),\alpha(T))]\\
&\quad = E^{txi}[I_{\{\tau^u>T\}}\mathrm{e}^{-\beta(T-t)}U(X^u(T),\alpha(T))].
\end{aligned} \tag{3.16}$$

Taking limit as $k\to\infty$ in (3.13) yields:

$$\begin{aligned}
W(t,x,i) &\geq E^{txi}\left[\int_t^{\tau^u\wedge T} \mathrm{e}^{-\beta(s-t)}U(c(s),\alpha(s))ds\right.\\
&\qquad \left. + I_{\{\tau^u>T\}}\mathrm{e}^{-\beta(T-t)}U(X^u(T),\alpha(T))\right]\\
&= J(t,x,i;u(\cdot)).
\end{aligned} \tag{3.17}$$

This proves Part (a) of the theorem.

For Part (b), note that when $u(\cdot) = u^*(\cdot)$ and $X^u(\cdot) = X^*(\cdot)$, the inequality (3.6) becomes an equality, that is,

$$\mathcal{L}^{u^*(s)}W(s,X^*(s),\alpha(s)) + U(c^*(s),\alpha(s)) = 0, \quad t\leq s\leq \tau^*. \tag{3.18}$$

Consequently, (3.13) becomes an equality when $u(\cdot)$ and $X^u(\cdot)$ are replaced by $u^*(\cdot)$ and $X^*(\cdot)$, respectively. Taking limit as $k \to \infty$, we obtain $W(t,x,i) = J(t,x,i;u^*(\cdot))$. Combining Parts (a) and (b), we indeed have:

$$W(t,x,i) = V(t,x,i) = J(t,x,i;u^*(\cdot)). \tag{3.19}$$

This completes the proof of Theorem 3.1. □

To apply the verification Theorem 3.1, we look for a solution $W(t,x,i)$ of the HJB (2.11), (2.12) that satisfies the required conditions in Theorem 3.1. Let

$$\Phi(t,x,i,\pi) = \pi^T[\mu(i) - r(i)\mathbb{1}]xW_x(t,x,i) + \frac{1}{2}\pi^T\Sigma(i)\pi x^2 W_{xx}(t,x,i). \tag{3.20}$$

Then under the condition that $W_{xx}(t,x,i) < 0$, we have

$$\hat{\pi} = \underset{\pi\in\mathbb{R}^n}{\operatorname{argsup}}\Phi(t,x,i,\pi) = -\frac{W_x(t,x,i)}{xW_{xx}(t,x,i)}\Sigma^{-1}(i)[\mu(i) - r(i)\mathbb{1}] \tag{3.21}$$

and

$$\hat{c} = \underset{c\geq 0}{\operatorname{argsup}}\{U(c,i) - cW_x(t,x,i)\} = (U'(W_x(t,x,i),i))^{-1}. \tag{3.22}$$

Note that the right-hand side of (3.22) denotes the inverse function of $U'(\cdot,i)$ evaluated at $W_x(t,x,i)$. Moreover,

$$\Phi(t,x,i,\hat{\pi}) = -\frac{W_x^2(t,x,i)}{2W_{xx}(t,x,i)}\nu(i), \tag{3.23}$$

where

$$\nu(i) = [\mu(i) - r(i)\mathbb{1}]^T\Sigma^{-1}(i)[\mu(i) - r(i)\mathbb{1}], \quad i = 1,\ldots,m_0. \tag{3.24}$$

Substituting π and c with $\hat{\pi}$ and $\hat{c}$ in (2.11), respectively, we obtain:

$$\begin{aligned} W_t(t,x,i) &- \frac{W_x^2(t,x,i)}{2W_{xx}(t,x,i)}\nu(i) + U(\hat{c},i) - \hat{c}W_x(t,x,i) + r(i)xW_x(t,x,i) \\ &- \beta W(t,x,i) + \sum_{j\neq i} q_{ij}[W(t,x,j) - W(t,x,i)] = 0, \quad i = 1,\ldots,m_0. \end{aligned} \tag{3.25}$$

Equations (3.21)–(3.25) will be used in the next section to find an explicit solution of the optimal control problem when a power utility function is considered.

4. Explicit Solution for Power Utility Function

In this section, we consider a power utility function given by

$$U(x,i) = \lambda_i x^\gamma, \quad 0 < \gamma < 1, \quad \lambda_i > 0. \tag{4.1}$$

Note that this utility function maintains the same power term x^γ in all regimes but changes the coefficients as the regime switches. It generalizes the one used in Sotomayor & Cadenillas (2009) where a specific value $\gamma = \frac{1}{2}$ is chosen for a two-regime infinite-horizon problem. Also note that for each $i \in \mathcal{M}$, $U(\cdot, i)$ belongs to the class of utility functions known as the HARA class.

We then look for a solution $W(t, x, i)$ of (2.11) in the form

$$W(t, x, i) = x^{\gamma}\phi(t, i) \tag{4.2}$$

for suitable functions $\phi(t, i)$, $i = 1, 2, \ldots, m_0$ to be determined. In this case, $W_x = \gamma x^{\gamma-1}\phi$, $W_{xx} = \gamma(\gamma-1)x^{\gamma-2}\phi$. It follows from (3.21) and (3.22) that

$$\hat{\pi} = \frac{1}{1-\gamma}\Sigma^{-1}(i)[\mu(i) - r(i)\mathbb{1}] \tag{4.3}$$

and

$$\hat{c} = \left(\frac{W_x(t, x, i)}{\lambda_i \gamma}\right)^{\frac{1}{\gamma-1}} = x\left(\frac{\phi(t, i)}{\lambda_i}\right)^{\frac{1}{\gamma-1}}. \tag{4.4}$$

Substituting (4.1)–(4.4) into (3.25), we obtain the following system of nonlinear ODEs for the functions $\phi(t, i)$, $i = 1, 2, \ldots, m_0$:

$$\begin{aligned} &\phi_t(t, i) + \left(\gamma r(i) + \frac{\gamma\nu(i)}{2(1-\gamma)} - \beta\right)\phi(t, i) + (1-\gamma)\lambda_i^{\frac{1}{1-\gamma}}\phi^{\frac{\gamma}{\gamma-1}}(t, i) \\ &\quad + \sum_{j\neq i} q_{ij}[\phi(t, j) - \phi(t, i)] = 0, \end{aligned} \tag{4.5}$$

with the terminal condition $\phi(T, i) = \lambda_i$.

Using the fixed point theorem, we can show the existence of a local (non-negative) solution ϕ of (4.5) on some interval (t_0, T). Given $t_0 < t < T$. For $t \le s \le T$, using the Itô's formula for Markovian-modulated processes (see Buffington & Elliott 2002), we have

$$d\phi(s, \alpha(s)) = \left(\phi_t(s, \alpha(s)) + \sum_{j\neq\alpha(s)} q_{\alpha(s)j}[\phi(s, j) - \phi(s, \alpha(s))]\right)ds + dM_\alpha(s), \tag{4.6}$$

where $M_\alpha(s)$, $t \le s \le T$ is a martingale.

Let

$$\Psi(s) = \exp\left(\int_t^s \left(\gamma r(\alpha(\theta)) + \frac{\gamma\nu(\alpha(\theta))}{2(1-\gamma)} - \beta\right)d\theta\right). \tag{4.7}$$

Using (4.5) and (4.6), we obtain:

$$d\left[\Psi(s)\phi(s, \alpha(s))\right] = \Psi(s)\left(-(1-\gamma)\lambda_{\alpha(s)}^{\frac{1}{1-\gamma}}\phi^{\frac{\gamma}{\gamma-1}}(s, \alpha(s)) + dM_\alpha(s)\right)$$

and then

$$\begin{aligned} \Psi(T)\phi(T, \alpha(T)) - \phi(t, \alpha(t)) = &-\int_t^T (1-\gamma)\Psi(s)\lambda_{\alpha(s)}^{\frac{1}{1-\gamma}}\phi^{\frac{\gamma}{\gamma-1}}(s, \alpha(s))ds \\ &+ \int_t^T \Psi(s)dM_\alpha(s). \end{aligned}$$

Taking expectation and using $\phi(T,\alpha(T)) = \lambda_{\alpha(T)}$ and $\alpha(t) = i$, we have:

$$\phi(t,i) = E^{ti}[\Psi(T)\lambda_{\alpha(T)}] + (1-\gamma)E^{ti}\left[\int_t^T \Psi(s)\lambda_{\alpha(s)}^{\frac{1}{1-\gamma}}\phi^{\frac{\gamma}{\gamma-1}}(s,\alpha(s))ds\right]. \tag{4.8}$$

Note that $\Psi(s)$ is bounded uniformly in t. It follows that $\phi(t,i) > C_0$ for some positive constant $C_0 > 0$. On the other hand, since $0 < \gamma < 1$, $\frac{\gamma}{\gamma-1} < 0$. This implies that $\phi^{\frac{\gamma}{\gamma-1}}(s,\alpha(s)) < (C_0)^{\frac{\gamma}{\gamma-1}}$, $t \le s \le T$. Using this result in (4.8) we can show that $\phi(t,i) < C^0$ for some constant $C^0 > 0$. This establishes both lower and upper bounds for $\phi(t,i)$. Standard arguments from the theory of nonlinear ODEs (see for example, Coddington & Levinson 1955) can show the following result.

Lemma 4.1. *The system of nonlinear ODEs* (4.5) *with the terminal condition* $\phi(T,i) = \lambda_i$, $i = 1,2,\ldots,m_0$ *admits a unique non-negative and continuously differentiable solution.*

Theorem 4.2. *For the power utility function* (4.1), *let the investment and consumption policy* $\pi^*(\cdot)$ *and* $c^*(\cdot)$ *be given by*:

$$\pi^*(s) = \frac{1}{1-\gamma}\Sigma^{-1}(\alpha(s))[\mu(\alpha(s)) - r(\alpha(s))\mathbb{1}], \tag{4.9}$$

and

$$c^*(s) = \left(\frac{\phi(s,\alpha(s))}{\lambda_{\alpha(s)}}\right)^{\frac{1}{\gamma-1}} X^*(s), \tag{4.10}$$

where $X^*(\cdot)$ *is the solution of the linear SDE*:

$$dX^*(s) = X^*(s)\left[\frac{\nu(\alpha(s))}{1-\gamma} + r(\alpha(s)) - \left(\frac{\phi(s,\alpha(s))}{\lambda_{\alpha(s)}}\right)^{\frac{1}{\gamma-1}}\right]dt + X^*(s)\frac{1}{1-\gamma}$$

$$\times [\mu(\alpha(s)) - r(\alpha(s))\mathbb{1}]^T\Sigma^{-1}(\alpha(s))\sigma(\alpha(s))dB(s), \quad X^*(t) = x, \tag{4.11}$$

where $\phi(t,i)$, $i = 1,2,\ldots,m_0$ *is the solution of* (4.5) *with the terminal condition* $\phi(T,i) = \lambda_i$, $i = 1,2,\ldots,m_0$. *Then*, $u^*(\cdot) = (\pi^{*,T}(\cdot), c^*(\cdot))^T$ *is an optimal control process to the optimization problem* (2.9), (2.10) *and the value function is given by* $V(t,x,i) = x^\gamma\phi(t,i)$.

Proof. We use the verification Theorem 3.1. First we show that $u^*(\cdot)$ is an admissible control, i.e. $u^*(\cdot) \in \mathcal{A}_{txi}$. Clearly, $u^*(\cdot)$ is $\mathbb{F}$-adapted. To check (2.5), note that

$$E^{txi}\left[\int_t^T |\pi^*(s)|^2 ds\right] = \frac{1}{(1-\gamma)^2}E^{txi}\left[\int_t^T |\Sigma^{-1}(\alpha(s))[\mu(\alpha(s)) - r(\alpha(s))\mathbb{1}]|^2 ds\right]$$

$$\le \frac{1}{(1-\gamma)^2}\max_{1\le i\le m_0}|\Sigma^{-1}(i)[\mu(i) - r(i)\mathbb{1}]|^2(T-t) < \infty. \tag{4.12}$$

Therefore (2.5) holds for $u^*(\cdot)$. Next, since $\phi(\cdot,i)$ is bounded on $[t,T]$ for each $i \in \mathcal{M}$, there exists a constant C such that $(\frac{\phi(s,\alpha(s))}{\lambda_{\alpha(s)}})^{\frac{1}{\gamma-1}} \le C$, $t \le s \le T$. Note that (4.11) is a linear SDE with bounded coefficients. Standard arguments on the estimation of moments of solutions to SDE can be used to show that (2.7) holds for $X^*(\cdot)$ (see for example, Krylov 1980, Chapter 2). Finally, we have $c^*(s) \geq 0$, $t \leq s \leq T$ and

$$E^{txi}\left[\int_t^T c^*(s)ds\right] = E^{txi}\left[\int_t^T \left(\frac{\phi(s,\alpha(s))}{\lambda_{\alpha(s)}}\right)^{\frac{1}{\gamma-1}} X^*(s)ds\right]$$
$$\leq CE^{txi}\left[\int_t^T X^*(s)ds\right] \leq C(T-t)E^{txi}\left[\sup_{t\leq s\leq T}|X^*(s)|\right] < \infty. \tag{4.13}$$

Hence (2.6) holds. Consequently, we have $u^*(\cdot) \in \mathcal{A}_{txi}$.

Clearly, the function $W(t,x,i) = x^\gamma\phi(t,i)$ is in the class $C^{1,2}((0,T)\times(0,\infty)) \cap C([0,T]\times[0,\infty))$ for each $i \in \mathcal{M}$, and satisfies the condition (3.2). Therefore, applying Theorem 3.1, we conclude that $u^*(\cdot)$ is an optimal control process to the optimization problem (2.9), (2.10) and the value function is given by $V(t,x,i) = x^\gamma\phi(t,i)$. □

Next, we consider an averaged problem for which the market parameters are replaced by their averages over all regimes. Assume that the Markov chain $\alpha(t)$ is strongly irreducible. Let $p = (p_1,\ldots,p_{m_0})$ denote its stationary distribution, i.e. the vector p is the unique solution of the system $pQ = 0$, $\sum_{i=1}^{m_0} p_i = 1$, $p_i > 0$ for $i = 1,\ldots,m_0$. Let

$$\bar{r} = \sum_{i=1}^{m_0} r(i)p_i, \quad \bar{\mu}_k = \sum_{i=1}^{m_0} \mu_k(i)p_i, \quad 1 \leq k \leq n \tag{4.14}$$

and

$$\bar{\sigma}_{kj}^2 = \sum_{i=1}^{m_0} \sigma_{kj}^2(i)p_i, \quad 1 \leq k \leq n, \ 1 \leq j \leq d. \tag{4.15}$$

Note that $\bar{r}$, $\bar{\mu}_k$ and $\bar{\sigma}_{kj}^2$ are respectively, the probabilistic averages of $r(\alpha(t))$, $\mu_k(\alpha(t))$, $\sigma_{jk}^2(\alpha(t))$ with respect to the stationary distribution p. Let $\bar{\mu} = (\bar{\mu}_1,\ldots,\bar{\mu}_n)^T$, $\bar{\sigma} = (\bar{\sigma}_{kj})_{n\times d}$. Replacing $r(\alpha(t))$, $\mu(\alpha(t))$, $\sigma(\alpha(t))$ with $\bar{r}$, $\bar{\mu}$, $\bar{\sigma}$, respectively in the investment and consumption model (2.4), we obtain the following wealth equation with constant coefficients:

$$dX(t) = \bar{r}X(t)dt + X(t)\pi^T(t)(\bar{\mu} - \bar{r}\mathbb{1})dt - c(t)dt + X(t)\pi^T(t)\bar{\sigma}dB(t). \tag{4.16}$$

Moreover, we use a power utility function defined by $\bar{U}(x) = \bar{\lambda}x^\gamma$ where

$$\bar{\lambda} = \sum_{i=1}^{m_0} \lambda_i p_i \tag{4.17}$$

is the average of $\lambda_{\alpha(t)}$ with respect to the stationary distribution p. Consider the averaged optimization problem (or the stationary problem) for which the objective functional $\bar{J}$ is defined by

$$\bar{J}(t,x,u(\cdot)) = E^{tx}\left[\int_t^{\tau^u \wedge T} \mathrm{e}^{-\beta(s-t)}\bar{U}(c(s))ds + I_{\{\tau^u > T\}}\mathrm{e}^{-\beta(T-t)}\bar{U}(X^u(T))\right] \tag{4.18}$$

and the value function $\bar{V}$ is defined by

$$\bar{V}(t,x) = \sup_{u(\cdot)\in\mathcal{A}_{tx}} \bar{J}(t,x,u(\cdot)), \tag{4.19}$$

where $\mathcal{A}_{tx}$ denotes the collection of admissible controls $u(\cdot)$ with respect to the initial time t and initial wealth $x(t) = x$. Note that this averaged problem is in virtual a classical Merton problem. In this case, the optimal solutions (4.3) and (4.4) become:

$$\hat{\pi} = \frac{1}{1-\gamma}\bar{\Sigma}^{-1}[\bar{\mu} - \bar{r}\mathbb{1}], \tag{4.20}$$

where $\bar{\Sigma} = \bar{\sigma}\bar{\sigma}^T$, and

$$\hat{c} = x\left(\frac{\bar{\phi}(t)}{\bar{\lambda}}\right)^{\frac{1}{\gamma-1}}, \tag{4.21}$$

where the function $\bar{\phi}(t)$ satisfies the ODE:

$$\bar{\phi}'(t) + \left(\gamma\bar{r} + \frac{\gamma\bar{\nu}}{2(1-\gamma)} - \beta\right)\bar{\phi}(t) + (1-\gamma)\bar{\lambda}^{\frac{1}{1-\gamma}}\bar{\phi}^{\frac{\gamma}{\gamma-1}}(t) = 0, \quad \bar{\phi}(T) = \bar{\lambda}, \tag{4.22}$$

where

$$\bar{\nu} = (\bar{\mu} - \bar{r}\mathbb{1})^T\bar{\Sigma}^{-1}(\bar{\mu} - \bar{r}\mathbb{1}). \tag{4.23}$$

The value function is given by $\bar{V}(t,x) = x^\gamma\bar{\phi}(t)$.

The ODE (4.22) can be easily solved. Let $\bar{\phi} = h^{1-\gamma}$. Then (4.22) is transformed into a first-order linear ODE for h:

$$h'(t) + Hh(t) + \bar{\lambda}^{\frac{1}{1-\gamma}} = 0, \quad h(T) = \bar{\lambda}^{\frac{1}{1-\gamma}}, \tag{4.24}$$

where

$$H = \frac{1}{1-\gamma}\left(\gamma\bar{r} + \frac{\gamma\bar{\nu}}{2(1-\gamma)} - \beta\right). \tag{4.25}$$

The solution of (4.24) is then given by:

$$h(t) = \begin{cases} \dfrac{1}{H}\bar{\lambda}^{\frac{1}{1-\gamma}}[-1 + (H+1)e^{H(T-t)}] & \text{if } H \neq 0, \\ \bar{\lambda}^{\frac{1}{1-\gamma}}(T+1-t) & \text{if } H = 0. \end{cases} \tag{4.26}$$

It can be easily shown that $h(t) > 0$ for $0 \le t \le T$. The value function is then given explicitly by

$$\bar{V}(t,x) = \begin{cases} x^{\gamma}\left(\dfrac{1}{H}\right)^{1-\gamma} \bar{\lambda}[-1 + (H+1)e^{H(T-t)}]^{1-\gamma} & \text{if } H \neq 0, \\ x^{\gamma}\bar{\lambda}(T+1-t)^{1-\gamma} & \text{if } H = 0. \end{cases} \tag{4.27}$$

In the next section, we will numerically compare the optimal investment and consumption policies with regime-switching to the averaged problem. This can help us get insight into the important role played by the Markov chain in making investment and consumption decisions.

5. Numerical Example

In this section, we provide a numerical example to illustrate the analytical results we obtained in Sec. 4 for the optimal investment and consumption policies in regime-switching models. In particular, we show how the optimal consumption ratios deviate from the averaged case (the Merton line) due to regime-switching. These results reveal the important role played by the Markov chain $\alpha(\cdot)$ in the investment and consumption models and can help us better understand the market behavior under different macroeconomic conditions.

We consider one risky asset and two market regimes, $n = 1$, $d = 1$ and $m_0 = 2$. In this case, the generator of the Markov chain $\alpha(\cdot)$ takes the special form of

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \quad q_{12} > 0, \quad q_{21} > 0,$$

where q_{12} is the jump rate from regime 1 to regime 2 and q_{21} is the jump rate from regime 2 to regime 1. The values of various model parameters used in the numerical study are specified in the following: the interest rate $r(1) = 0.05$, $r(2) = 0.02$, the stock appreciation rates $\mu(1) = 0.10$, $\mu(2) = 0.15$, the stock volatilities $\sigma(1) = 0.35$, $\sigma(2) = 0.75$, the discount factor $\beta = 0.1$, the utility functions $U(x,1) = 3x^{0.5}$, $U(x,2) = 2x^{0.5}$ (that is, $\gamma = 0.5$, $\lambda_1 = 3$, $\lambda_2 = 2$), the time horizon is $T = 1$ (year). The values for r, μ and σ satisfy $0 < \mu(1) - r(1) < \mu(2) - r(2)$, $0 < \sigma(1) < \sigma(2)$, and $\frac{\mu(2)-r(2)}{\sigma^2(2)} < \frac{\mu(1)-r(1)}{\sigma^2(1)}$. Hence regime 1 can be considered as a bull market and regime 2 a bear market. See Sotomayor & Cadenillas (2009, Sec. 5) for discussions. From (4.3) we have, for this example, $\hat{\pi} = \frac{1}{1-\gamma}\frac{\mu(i)-r(i)}{\sigma^2(i)}$, $i = 1, 2$. Thus the investor should invest a larger fraction of his/her wealth in stock in regime 1 than in regime 2. Note that (4.3) indicates that the optimal fraction of wealth in stock does not depend on time t but does depend on the market regime. The optimal fraction is constant in each regime and changes to a different constant if a regime-switching occurs. This observation is the same as that for the infinite-horizon problem treated in Sotomayor & Cadenillas (2009).

In view of (4.4), we let

$$\theta(t,i) = \left(\frac{\phi(t,i)}{\lambda_i}\right)^{\frac{1}{\gamma-1}}, \quad i \in \mathcal{M}$$

denote the optimal consumption ratio in regime i, $i \in \mathcal{M}$. Then the optimal consumption rate $\hat{c}$ is given by $\hat{c} = \theta(t,i)x$. Similarly, let

$$\bar{\theta}(t) = \left(\frac{\bar{\phi}(t)}{\bar{\lambda}}\right)^{\frac{1}{\gamma-1}}$$

be the optimal consumption ratio for the averaged problem (4.18) and (4.19). Then the optimal consumption rate for the averaged problem is given by $\hat{c} = \bar{\theta}(t)x$ in view of (4.21). Note that the optimal consumption ratio depends not only on the regime, but also on the time t. This is not the case for the infinite-horizon problem (Sotomayor & Cadenillas 2009) for which the optimal consumption ratio was shown only dependent on the regime. In the numerical study, we numerically solve the ODE system (4.5) for the functions $\phi(t,i)$, $i = 1,2$ and then plot $\theta(t,i)$ for the two different regimes. For comparison, we also plot $\bar{\theta}(t)$ for the averaged problem (the Merton line).

For the Markov chain $\alpha(\cdot)$, we choose $q_{12} = 3$ and $q_{21} = 4$. That means on average the market switches three times per year from regime 1 to regime 2 and four times from regime 2 to regime 1. The stationary distribution of the Markov chain $\alpha(\cdot)$ is $p = (\frac{4}{7}, \frac{3}{7})$. Figure 1 shows the three ratio functions $\theta(t,1)$, $\theta(t,2)$, and $\bar{\theta}(t)$,

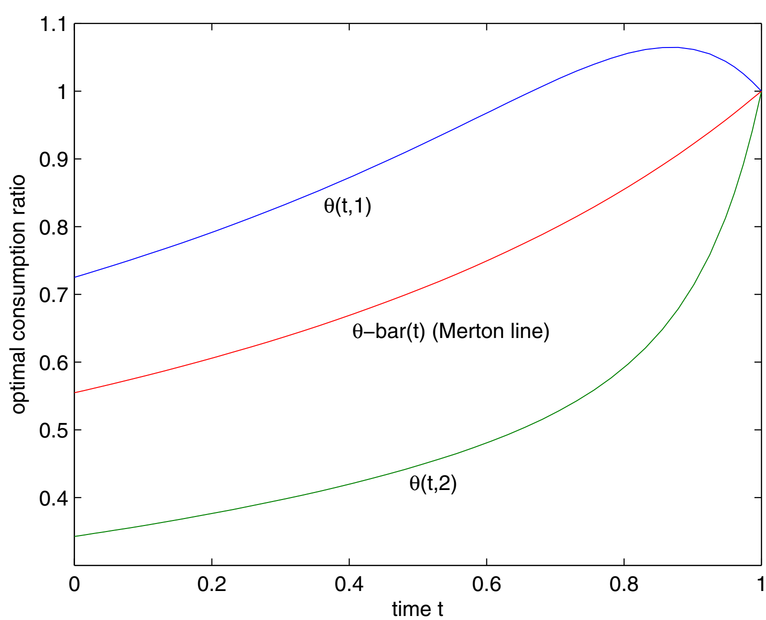

Fig. 1. Optimal consumption ratios for the two regimes and for the averaged case.

$0 \le t \le T$. We observe a number of interesting facts from Fig. 1. First, we see that $\theta(t,1) > \bar{\theta}(t) > \theta(t,2)$ for all time $t < T$ and all three curves approach 1 at the maturity time $t = T$. This indicates that the investor consumes his/her wealth at a higher ratio in the bull market (regime 1) and a lower ratio in the bear market (regime 2). The consumption ratio for the averaged problem is in between the two for the two different regimes, as expected. Second, $\theta(t,2)$ is an increasing function of the time t. Thus the investor consumes at a low ratio at the beginning, gradually increases this ratio and eventually consumes all of his/her wealth at the maturity $t = T$. This is consistent with the Merton line $\bar{\theta}(t)$ that also increases as the time increases. On the other hand, we see a different pattern for $\theta(t,1)$. The investor starts with a relative low ratio, increases this ratio to its maximum point at around $t = 0.88$, and then decreases it gradually to 1 at $t = T$. It should be pointed out that due to random switching between the two regimes, the investor should change the consumption ratio accordingly. Assume the current market is bull and the investor consumes at a higher ratio. If a regime-switching (from bull to bear) occurs, then it is necessary for the investor to reduce the ratio immediately according to $\theta(t,2)$. Conversely, the investor should increases his/her ratio by following $\theta(t,1)$ if the market switches from bear to bull. Hence, overall the consumption behavior is neither increasing nor decreasing, and is governed by the regime-switching introduced in the model.

The optimal fraction of wealth in stock $\hat{\pi}$ is calculated to be 81.63%, 46.22%, and 54.19% for regime 1, regime 2 and the averaged problem, respectively. That means when the market is in regime 1, the investor should maintain 81.63% of his/her wealth in the stock investment. Once a regime-switching (from regimes 1 to 2) occurs, he/she should change this percentage to 46.22%. Similarly, adjustments need to be made when the market switches from regime 2 to regime 1.

6. Concluding Remarks

In this paper, we applied the methods of stochastic optimal control to solve a finite-horizon optimal investment and consumption problem in continuous-time regime-switching models. We extended the results of the classical Merton problem to the regime-switching case. It has been well documented in the literature that including a regime-switching component in the market models can better describe the market changes under different macroeconomic conditions. Therefore, the results obtained in this paper will help us better understand the complicated behavior of the optimal dynamics for the investment and consumption problems. An interesting project for the next-step research is to consider an unobservable Markov chain in our problem setup which is different from the works mentioned in the introduction.

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