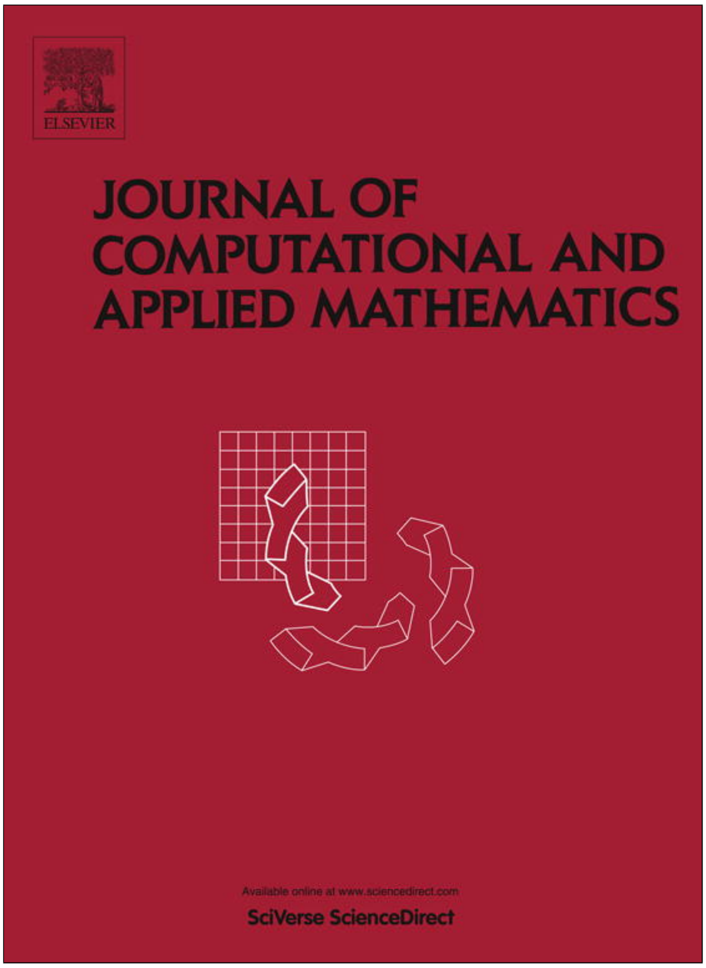

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A lattice method for option pricing with two underlying assets in the regime-switching model

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ABSTRACT

In this work we develop an efficient lattice approach for option pricing with two underlying assets whose prices are governed by regime-switching models. Jump amplitudes are specified in a way such that the lattice achieves complete node recombination along each asset variable and grows quadratically as the number of time steps increases. Jump probabilities are obtained by solving a related quadratic programming problem. The weak convergence of the discrete lattice approximations to the continuous-time regime-switching diffusion processes is established. The lattice is employed to price both European and American options written on the maximum and minimum of two assets in different regimes. Numerical results are provided and compared for the European options with a Monte-Carlo simulation approach.

1. Introduction

Lattice has been an important approach in computational finance ever since the innovative development of the binomial tree by Cox, Ross and Rubinstein [1] for pricing options on a single asset. A number of papers have extended the CRR binomial tree method to multiple assets and proposed lattice methods for pricing multivariate contingent claims. See [2–4], among others. Note that in these papers, the asset price is assumed to follow a constant geometric Brownian motion (GBM) model and the jump amplitudes of the trees remain constant as the time evolves.

Regime-switching models have drawn considerable attention in recent years in financial mathematics and computational finance, due to their capability of modeling non-constant and perhaps random market parameters (e.g. volatility and interest rate) and their comparably inexpensive computation. In this setup, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents randomly changing economical factors. Model parameters (drift and volatility coefficients) are assumed to depend on the Markov chain and are allowed to take different values in different regimes. As a result, both continuous dynamics and discrete events are present in the regime-switching model. Regime-switching models have been used in various subjects in financial mathematics including equity options [5–18], bond prices and interest rate derivatives [19–21], energy and commodity derivatives [22,21,23], portfolio selection [24], trading rules [25–29], and others. We refer the reader to Yin and Zhu [30] on the recent studies of regime-switching systems.

Developments of lattice approaches for valuing options on a single asset using regime-switching models have been presented in a number of articles. See [5,6,10,14,21,17,18], among others. Aingworth et al. [5] and Guo [10] extend the

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CRR binomial tree [1] in a straightforward way to a regime-switching geometric Brownian motion (GBM) model, resulting in a partially recombining tree. One disadvantage of their tree constructions is that the number of nodes can grow too fast as the number of time steps increases. According to Nelson and Ramaswamy [31], a lattice approximation to a (one-dimensional) diffusion is "computationally simple" if the number of nodes grows at most linearly in the number of time steps. To reduce the growth rate of nodes, Bollen [6] considers a GBM model with two regimes and proposes a pentanomial tree that achieves complete node recombination and grows linearly. Liu [14] adapts a similar idea of [6] and develops a linear tree for the regime-switching GBM model allowing any number of regimes. It is numerically demonstrated in [14] that the new recombining tree is efficient and accurate in pricing both European and American options with a single asset. Liu [21] extends the tree method [14] to a class of regime-switching mean-reverting models that have been frequently used for stochastic interest rates, energy and commodity prices. Yuen and Yang [17,18] develop a trinomial tree method for option pricing in Markov regime-switching models and use the proposed tree to price Asian options and equity-indexed annuities.

In this work we extend the approach of [14] to options with two underlying assets whose prices are governed by the regime-switching GBM models. In constructing the lattice, jump amplitudes for the two variables are specified in a way such that the lattice achieves complete node recombination along each variable and as a result, the two-dimensional lattice grows quadratically as the number of time steps increases. Jump probabilities are determined by solving a related quadratic programming problem. In addition to the algorithm, by following a method of martingale problem formulation, we establish the weak convergence of the discrete lattice approximations to the continuous-time regime-switching GBM processes. We employ the new lattice method to price both European and American options written on the maximum and minimum of two assets in different regimes and report numerical results. For the European options, we also numerically compare our lattice method with a Monte-Carlo simulation approach.

The paper is organized as follows. Section 2 is devoted to the lattice development. The regime-switching model is introduced first. The details of constructing the lattice are given next. Section 3 is concerned with the weak convergence of the lattice approximations to the continuous-time processes. Section 4 is concerned with the numerical study. The lattice approach is employed to price both European and American options written on the maximum and minimum of two assets in different regimes. For the European options, the results are compared with a Monte-Carlo simulation approach. Section 5 provides further remarks and concludes the paper.

2. Regime-switching lattice with two assets

We consider a continuous-time Markov chain α_t that takes a value at a time among m_0 different states, where m_0 is the total number of states considered for the model. Each state represents a particular regime and is labeled by an integer i between 1 and m_0. Hence the state space of α_t is the set $\mathcal{M} := \{1, \ldots, m_0\}$. For example, if $m_0 = 2$ (two regimes), then $\alpha_t = 1$ may indicate a bullish market and $\alpha_t = 2$ a bearish market. In this paper we assume that the Markov chain α_t is observable.

We note that introducing a Markov chain into the model results in an incomplete market [32,10]. It has been known that the absence of arbitrage opportunities is equivalent to the existence of an equivalent Martingale measure. However, the equivalent Martingale measure is not unique if the market is incomplete. Elliott, Chan and Siu [32] and Siu [33] employ a regime-switching random Esscher transform to choose an equivalent Martingale measure for pricing derivatives. We decide not to include the argument in this paper and instead assume that the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given. Another related and interesting issue is whether one should price the regime-switching risk. We refer the readers to Elliott, Siu and Badescu [34], Siu [35] and Siu and Yang [36] for more discussions.

In this work we consider options written on two correlated underlying risky assets whose prices $S_1(t)$ and $S_2(t)$ are given by the following regime-switching geometric Brownian motions. For $i = 1, 2$,

$$\frac{dS_i(t)}{S_i(t)} = r(\alpha_t)dt + \sigma_i(\alpha_t)dB_i(t), \quad t \geq 0, \tag{2.1}$$

where $\sigma_i(\alpha_t)$ is the volatility rate of the ith asset, $r(\alpha_t)$ is the risk-free interest rate, $B_1(t)$ and $B_2(t)$ are two correlated Brownian motions with $dB_1(t)dB_2(t) = \rho dt$ where $-1 < \rho < 1$ is the correlation coefficient between the two Brownian motions. We assume that $B_1(t)$ and $B_2(t)$ are independent of the Markov chain α_t. Note that $\sigma_i(\alpha_t)$ and $r(\alpha_t)$ depend on the Markov chain α_t, indicating that they can take different values in different regimes. We assume that $\sigma_i(\alpha) > 0$, $\alpha = 1, \ldots, m_0, i = 1, 2$.

Remark 1. We could assume that the correlation coefficient ρ depends on the market regime as well in our lattice model. However, noting that (2.1) implies $\frac{dS_1(t)}{S_1(t)}\frac{dS_2(t)}{S_2(t)} = \sigma_1(\alpha_t)\sigma_2(\alpha_t)\rho dt$, we see that the regime-dependent feature for the correlation between the two risky asset returns is captured by assuming that the volatility rates depend on the market regime. For this reason we choose to use a constant correlation coefficient ρ in the exposition of our lattice construction.

To proceed, let $X_i(t) = \ln(S_i(t)/S_i(0))$. Then $S_i(t) = S_i(0)e^{X_i(t)}$ where $X_i(t)$, $i = 1, 2$ are the solutions of the stochastic differential equations:

$$dX_i(t) = a_i(\alpha_t)dt + \sigma_i(\alpha_t)dB_i(t), \quad X_i(0) = 0, \tag{2.2}$$

where

$$a_i(\alpha_t) = r(\alpha_t) - \frac{1}{2}\sigma_i^2(\alpha_t). \tag{2.3}$$

In what follows we construct a two-dimensional lattice for the joint process $(X_1(t), X_2(t), \alpha_t)$. The corresponding approximations for the asset prices $S_1(t), S_2(t)$ can be easily obtained from the lattice approximations of $X_1(t), X_2(t)$. Let $T > 0$ be the option maturity. Divide the option life $[0, T]$ into N steps and let $h = \frac{T}{N}$ be the time step size. Let $(X_{1,k}, X_{2,k}, \alpha_k) := (X_1(t), X_2(t), \alpha_t)_{t=kh}$ be the state of the lattice at the kth step, $k = 0, 1, \dots, N$. We use three moves for each of the two state variables $X_{1,k}$ and $X_{2,k}$, namely, an up move, a down move, and a middle stay (no move). Choose $\bar{\sigma}_1 > 0$ and $\bar{\sigma}_2 > 0$ for the sizes of one upward move for $X_{1,k}$ and $X_{2,k}$, respectively. Note that $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are independent of the regimes. Assume that $(X_{1,k}, X_{2,k}, \alpha_k) = (x_1, x_2, i)$. Let $l_{1,i}$ be the number of upward moves[1] for $X_{1,k+1}$ and $l_{2,i}$ be the number of upward moves for $X_{2,k+1}$ when the current regime is $\alpha_k = i$. Then, the three moves for $X_{1,k+1}$ are $x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}$ (up move), x_1 (no move) and $x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}$ (down move). Similarly, the three moves for $X_{2,k+1}$ are $x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}$ (up move), x_2 (no move) and $x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}$ (down move). Combining the two state variables together, we have the following nine nodes for $(X_{1,k+1}, X_{2,k+1})$ emanating from $(X_{1,k}, X_{2,k})$ (note that at this moment, the change of regime is not taken into account yet):

$$(X_{1,k+1}, X_{2,k+1}) = \begin{cases} (x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}), & \text{up, up} \\ (x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2), & \text{up, middle} \\ (x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}), & \text{up, down} \\ (x_1, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}), & \text{middle, up} \\ (x_1, x_2), & \text{middle, middle} \\ (x_1, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}), & \text{middle, down} \\ (x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}), & \text{down, up} \\ (x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2), & \text{down, middle} \\ (x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}), & \text{down, down.} \end{cases} \tag{2.4}$$

Let $p_{i,uu}, p_{i,um}, p_{i,ud}, p_{i,mu}, p_{i,mm}, p_{i,md}, p_{i,du}, p_{i,dm}, p_{i,dd}$ denote the conditional probabilities of the nine branches. By matching the means, variances and covariance implied by the lattice to those implied by the SDE (2.2), necessarily, we have

$$(p_{i,uu} + p_{i,um} + p_{i,ud})(l_{1,i}\bar{\sigma}_1\sqrt{h}) - (p_{i,du} + p_{i,dm} + p_{i,dd})(l_{1,i}\bar{\sigma}_1\sqrt{h}) = a_1(i)h, \tag{2.5}$$

$$(p_{i,uu} + p_{i,um} + p_{i,ud})(l_{1,i}\bar{\sigma}_1\sqrt{h})^2 + (p_{i,du} + p_{i,dm} + p_{i,dd})(l_{1,i}\bar{\sigma}_1\sqrt{h})^2 = a_1^2(i)h^2 + \sigma_1^2(i)h, \tag{2.6}$$

$$(p_{i,uu} + p_{i,mu} + p_{i,du})(l_{2,i}\bar{\sigma}_2\sqrt{h}) - (p_{i,ud} + p_{i,md} + p_{i,dd})(l_{2,i}\bar{\sigma}_2\sqrt{h}) = a_2(i)h, \tag{2.7}$$

$$(p_{i,uu} + p_{i,mu} + p_{i,du})(l_{2,i}\bar{\sigma}_2\sqrt{h})^2 + (p_{i,ud} + p_{i,md} + p_{i,dd})(l_{2,i}\bar{\sigma}_2\sqrt{h})^2 = a_2^2(i)h^2 + \sigma_2^2(i)h, \tag{2.8}$$

$$p_{i,uu}l_{1,i}l_{2,i}\bar{\sigma}_1\bar{\sigma}_2 h + p_{i,dd}l_{1,i}l_{2,i}\bar{\sigma}_1\bar{\sigma}_2 h - (p_{i,ud} + p_{i,du})l_{1,i}l_{2,i}\bar{\sigma}_1\bar{\sigma}_2 h = a_1(i)a_2(i)h^2 + \rho\sigma_1(i)\sigma_2(i)h, \tag{2.9}$$

$$p_{i,uu} + p_{i,um} + p_{i,ud} + p_{i,mu} + p_{i,\ mm} + p_{i,md} + p_{i,du} + p_{i,dm} + p_{i,dd} = 1. \tag{2.10}$$

By using (2.5), (2.6), we obtain

$$p_{i,uu} + p_{i,um} + p_{i,ud} = \frac{\sigma_1^2(i) + a_1(i)l_{1,i}\bar{\sigma}_1\sqrt{h} + a_1^2(i)h}{2(l_{1,i}\bar{\sigma}_1)^2}, \tag{2.11}$$

$$p_{i,du} + p_{i,dm} + p_{i,dd} = \frac{\sigma_1^2(i) - a_1(i)l_{1,i}\bar{\sigma}_1\sqrt{h} + a_1^2(i)h}{2(l_{1,i}\bar{\sigma}_1)^2}, \tag{2.12}$$

and by using (2.7), (2.8), we have

$$p_{i,uu} + p_{i,mu} + p_{i,du} = \frac{\sigma_2^2(i) + a_2(i)l_{2,i}\bar{\sigma}_2\sqrt{h} + a_2^2(i)h}{2(l_{2,i}\bar{\sigma}_2)^2}, \tag{2.13}$$

$$p_{i,ud} + p_{i,md} + p_{i,dd} = \frac{\sigma_2^2(i) - a_2(i)l_{2,i}\bar{\sigma}_2\sqrt{h} + a_2^2(i)h}{2(l_{2,i}\bar{\sigma}_2)^2}. \tag{2.14}$$

In addition, the following constraints are imposed:

$$0 \le p_{i,xy} \le 1, \quad x, y = u, m, d. \tag{2.15}$$

[1] We refer the readers to Liu [14, Proposition 2.1] for what possible values $l_{1,i}$ and $l_{2,i}$ can take.

On the other hand, the Markov chain α_{k+1} may stay at the state i with probability p^{α}_{ii} or jump to any other state $j \neq i$ with probability p^{α}_{ij}, where the one-step transition probabilities p^{α}_{ij} of the Markov chain α_k are defined by

$$p^{\alpha}_{ij} = P\{\alpha_{k+1} = j | \alpha_k = i\}, \quad 1 \le i, j \le m_0. \tag{2.16}$$

Note that those probabilities p^{α}_{ij} can be approximated in an appropriate way from the given distribution of the continuous-time Markov chain α_t. See (3.3) and (3.4) in the next section. Consequently, emanating from the node $(X_{1,k}, X_{2,k}, \alpha_k) = (x_1, x_2, i)$ at the current time step, there are $9m_0$ nodes for $(X_{1,k+1}, X_{2,k+1}, \alpha_{k+1})$ at the next time step, given by, for $j = 1, \ldots, m_0$,

$$(X_{1,k+1}, X_{2,k+1}, \alpha_{k+1}) = \begin{cases} (x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij} p_{i,uu} \\ (x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2, j), & \text{with probability } p^{\alpha}_{ij} p_{i,um} \\ (x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij} p_{i,ud} \\ (x_1, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij} p_{i,mu} \\ (x_1, x_2, j), & \text{with probability } p^{\alpha}_{ij} p_{i,mm} \\ (x_1, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij} p_{i,md} \\ (x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij} p_{i,du} \\ (x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2, j), & \text{with probability } p^{\alpha}_{ij} p_{i,dm} \\ (x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}, j), & \text{with probability } p^{\alpha}_{ij} p_{i,dd}. \end{cases} \tag{2.17}$$

Let $l_1 = \max_{1 \le i \le m_0} l_{1,i}$, $l_2 = \max_{1 \le i \le m_0} l_{2,i}$. It can be seen that at time step k, the number of possible different values for $X_{1,k}$ is at most $2kl_1 + 1$, and the number of possible different values for $X_{2,k}$ is at most $2kl_2 + 1$. Hence there are at most $(2kl_1 + 1)(2kl_2 + 1)$ different values for $(X_{1,k}, X_{2,k})$ and consequently at most $m_0(2kl_1 + 1)(2kl_2 + 1)$ different values for $(X_{1,k}, X_{2,k}, \alpha_k)$. This shows that our lattice grows at most quadratically as the time step k increases.

To obtain a set of conditional probabilities $p_{i,xy}$, $x, y = u, m, d$, we need to solve the system of linear equations (2.9)–(2.14) subject to the constraint (2.15). In view of Liu [14, Proposition 2.1], as long as we choose $\bar{\sigma}_1, \bar{\sigma}_2, l_{1,i}, l_{2,i}, i = 1, \ldots, m_0$ such that $l_{1,i}\bar{\sigma}_1 > \sigma_1(i)$ and $l_{2,i}\bar{\sigma}_2 > \sigma_2(i)$, $i = 1, \ldots, m_0$, then the RHS of (2.11)–(2.14) give feasible probability values, provided that the time step h is sufficiently small. To proceed, we make the following assumption.

Assumption 1. Assume that $l_{1,i}\bar{\sigma}_1 > \sigma_1(i)$ and $l_{2,i}\bar{\sigma}_2 > \sigma_2(i)$, $i = 1, \ldots, m_0$. Assume that there exists a positive number $h_0 > 0$ such that the system (2.9)–(2.15) has a feasible solution for all $h < h_0$.

We note that it may be possible to apply the linear programming results (in particular, Farkas's Lemma, see for instance, Ben-Tal and Nemirovski [37]) to find an upper bound h_0 in Assumption 1, which will be investigated in the future. On the other hand, our numerical computations find that Assumption 1 is satisfied by the used step sizes h in all numerical examples we have conducted.

As noted in the literature (see, for example Boyle [2]), good convergence results are obtained when the branch probabilities are roughly equal. Hence to determine the conditional probabilities $p_{i,xy}$, $x, y = u, m, d$, we consider the following optimization problem:

$$\min_{p_{i,xy}, x,y=u,m,d} \sum_{x,y=u,m,d} \left(p_{i,xy} - \frac{1}{9}\right)^2, \quad \text{subject to (2.9)–(2.15)}. \tag{2.18}$$

Note that in this paper we assume that the Markov chain α_t is independent of the Brownian motions $B_1(t)$ and $B_2(t)$. The transition probabilities of α_t do not enter Eqs. (2.9)–(2.15) that are used to determine the branch probabilities $p_{i,xy}$, $x, y = u, m, d$. We believe that a good selection of the branch probabilities combined with a good approximation of the Markov transition probabilities can produce good convergence of the discrete lattice. It will be an interesting future topic to investigate the relationship between the convergence of the lattice and the selection of branch probabilities if the Markov chain α_t is state-dependent. See Yin and Zhu [30] for recent developments of state-dependent regime-switching models and applications.

Standard techniques can be employed to solve the problem (2.18). See, for example, the book [38] by Nocedal and Wright. We also refer the readers to [38] for the existence and uniqueness of the solution of optimization problems like (2.18). For our numerical experiments presented in Section 4, we have used the MatLab function Quadprog (for solving quadratic programming problems) to solve (2.18) and obtained the branch probabilities $p_{i,xy}$, $x, y = u, m, d$.

Using the lattice we have constructed for the joint process $(X_1(t), X_2(t), \alpha_t)$, both European and American options with two underlying assets $S_1(t)$ and $S_2(t)$ can be valued following a similar procedure that is used in the CRR binomial tree for valuing options for a single asset in the standard Black–Scholes model, i.e., by starting at the last time step of the lattice $(n = N)$ and working backward iteratively. At each time step and for each node, we first calculate the probability weighted average of all option values at the nodes in the next time step that are directly emanated from the current node. We then discount the averaged future value using the interest rate at the current node. This gives us the discounted expectation of the future option value, where the expectation is taken with respect to the risk-neutral probabilities. For American options, we need to in addition check the early exercise possibility by comparing this value with the immediate exercise value. Because of the presence of regime-switching, those computations become more involved and special care needs to be taken. Note that the option values are functions of the underlying asset prices and also depend on the regime. Specifically, let non-negative $g(s_1, s_2)$ be the payoff function of a contingent claim, and let $V^k(x_1, x_2, i)$ denote the option value at time step k at the node associated with the state $(X_{1,k}, X_{2,k}, \alpha_k) = (x_1, x_2, i)$. Then, for the terminal step $k = N$,

$$V^N(x_1, x_2, i) = g(S_1(0)e^{x_1}, S_2(0)e^{x_2}), \quad i = 1, \dots, m_0. \tag{2.19}$$

At step $k < N$, first calculating

$$\begin{aligned} V_0^k(x_1, x_2, i) = e^{-r(i)h} \sum_{j=1}^{m_0} p_{ij}^{\alpha} \Big[& V^{k+1}(x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}, j)p_{i,uu} \\ & + V^{k+1}(x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2, j)p_{i,um} + V^{k+1}(x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}, j)p_{i,ud} \\ & + V^{k+1}(x_1, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}, j)p_{i,mu} + V^{k+1}(x_1, x_2, j)p_{i,\,mm} \\ & + V^{k+1}(x_1, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}, j)p_{i,md} + V^{k+1}(x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}, j)p_{i,du} \\ & + V^{k+1}(x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2, j)p_{i,dm} + V^{k+1}(x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}, x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}, j)p_{i,dd} \Big], \end{aligned} \tag{2.20}$$

then for European option, set $V^k(x_1, x_2, i) = V_0^k(x_1, x_2, i)$, and for American option,

$$V^k(x_1, x_2, i) = \max\Big\{ g(S_1(0)e^{x_1}, S_2(0)e^{x_2}), V_0^k(x_1, x_2, i) \Big\}. \tag{2.21}$$

3. Weak convergence of the lattice approximation

Numerical approximations of continuous-time switching diffusions are studied recently in, for example, Yin and Zhu [30]. In this section we mainly apply the methods in [30, Chapter 5] to establish the weak convergence of the discrete approximation processes implied by the recombining lattice to the continuous-time diffusion process (2.2). We will present the main results without proof details.

Fix $T > 0$. We consider the lattice construction for the time period $[0, T]$. Let $h > 0$ be the time step size. Then $N = \lfloor \frac{T}{h} \rfloor$ is the number of steps in the lattice, where $\lfloor z \rfloor$ denotes the floor of the real number z. Let the initial regime α_0 be given. For each fixed $h > 0$, we construct a continuous-time process $(X_1^h(t), X_2^h(t), \alpha^h(t))$, $t \ge 0$, such that

$$X_1^h(0) = X_1(0) = 0, X_2^h(0) = X_2(0) = 0, \alpha^h(0) = \alpha_0, \tag{3.1}$$

and

$$(X_1^h(t), X_2^h(t), \alpha^h(t)) = (X_{1,n}, X_{2,n}, \alpha_n), \tag{3.2}$$

for $nh \le t < (n+1)h$, $n = 0, 1, \dots, N-1$, where for given $(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)$, $(X_{1,n+1}, X_{2,n+1}, \alpha_{n+1})$ is determined as follows.

Let the generator $Q = (q_{ij})_{m_0 \times m_0}$ of the continuous-time Markov chain α_t be given. Then for the discrete Markov chain $\{\alpha_n, n \ge 0\}$, its probability transition matrix can be approximated by

$$P^h = I + hQ, \tag{3.3}$$

that implies

$$P\{\alpha_{n+1} = j | \alpha_n = i\} = \delta_{ij} + hq_{ij}, \quad 1 \le i, j \le m_0, \ 0 \le n \le N - 1, \tag{3.4}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$. It can be shown by applying [39, Theorem 3.1] that as $h \to 0$, the interpolated process $\alpha^h(\cdot)$ converges weakly to the continuous-time Markov chain α_t generated by Q. Note that (3.3) is a first-order approximation. Higher order approximations can be used in numerical experiments to improve convergence.

Next, for notational convenience, let $X_1^{h,u}(x_1, i) := x_1 + l_{1,i}\bar{\sigma}_1\sqrt{h}$, $X_1^{h,m}(x_1, i) := x_1$ and $X_1^{h,d}(x_1, i) := x_1 - l_{1,i}\bar{\sigma}_1\sqrt{h}$ denote the three values for $X_{1,n+1}$ to take at time step $n + 1$. Similarly, let $X_2^{h,u}(x_2, i) := x_2 + l_{2,i}\bar{\sigma}_2\sqrt{h}$, $X_2^{h,m}(x_2, i) := x_2$ and $X_2^{h,d}(x_2, i) := x_2 - l_{2,i}\bar{\sigma}_2\sqrt{h}$ denote the three values for $X_{2,n+1}$ to take at time step $n + 1$. Then the nine values for the joint variables $(X_{1,n+1}, X_{2,n+1})$ to take at time step $n + 1$ are given by $(X_1^{h,k}(x_1, i), X_2^{h,l}(x_2, i))$, $k, l = u, m, d$. We recall that $p_{i,kl}$, $k, l = u, m, d$ specify the conditional probabilities for the nine branches, namely,

$$P\left\{ (X_{1,n+1}, X_{2,n+1}) = (X_1^{h,k}(x_1, i), X_2^{h,l}(x_2, i)) \Big| (X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i) \right\} = p_{i,kl}, \tag{3.5}$$

for $k, l = u, m, d$. Moreover, Eqs. (2.5)–(2.9) imply that, for $i \in \mathcal{M}, x_1, x_2 \in \mathbb{R}$,

$$E\left[X_{1,n+1}\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = x_1 + a_1(i)h, \tag{3.6}$$

$$E\left[X_{2,n+1}\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = x_2 + a_2(i)h, \tag{3.7}$$

$$E\left[(X_{1,n+1} - x_1)^2\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = a_1^2(i)h^2 + \sigma_1^2(i)h, \tag{3.8}$$

$$E\left[(X_{2,n+1} - x_2)^2\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = a_2^2(i)h^2 + \sigma_2^2(i)h, \tag{3.9}$$

and

$$E\left[(X_{1,n+1} - x_1)(X_{2,n+1} - x_2)\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = a_1(i)a_2(i)h^2 + \rho\sigma_1(i)\sigma_2(i)h. \tag{3.10}$$

Using (3.6)–(3.9) and applying a discrete version of Gronwall's inequality, we can establish in the following lemma an upper bound of the second moments of $X_{1,n}, X_{2,n}, 0 \le n \le \lfloor \frac{T}{h} \rfloor$.

Lemma 1. *Given $T > 0$. Under Assumption 1, we have*

$$\sup_{0 \le n \le \lfloor T/h \rfloor} E\left[X_{j,n}^2\right] \le CTe^{CT} < \infty, \quad j = 1, 2, \tag{3.11}$$

where C is a positive constant.

We next establish the tightness of the continuous-time processes $\{(X_1^h(t), X_2^h(t)) : 0 \le t \le T\}$ in a suitable function space. To this end, we rewrite approximation (3.5) into the following iterative form:

$$X_{j,n+1} = X_{j,n} + a_j(\alpha_n)h + \xi_j(X_{j,n}, \alpha_n), \quad j = 1, 2, \tag{3.12}$$

where the random functions $\xi_j(\cdot, \cdot), j = 1, 2$ are defined as follows. For each $x \in \mathbb{R}$ and each $i \in \mathcal{M}$, $\xi_j(x, i)$ takes the three values $X_j^{h,k}(x, i) - x - a_j(i)h$, $k = u, m, d$, with the joint probabilities given by $p_{i,kl}$, $k, l = u, m, d$, i.e.,

$$P\left\{(\xi_1(x_1, i), \xi_2(x_2, i)) = (X_1^{h,k}(x_1, i) - x_1 - a_1(i)h, X_2^{h,l}(x_2, i) - x_2 - a_2(i)h)\right\} = p_{i,kl}, \tag{3.13}$$

for $k, l = u, m, d$. It then follows from (3.6)–(3.10) that

$$E\left[\xi_j(X_{j,n}, \alpha_n)\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = 0, \quad j = 1, 2, \tag{3.14}$$

$$E\left[\xi_j^2(X_{j,n}, \alpha_n)\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = \sigma_j^2(i)h, \quad j = 1, 2, \tag{3.15}$$

and

$$E\left[\xi_1(X_{1,n}, \alpha_n)\xi_2(X_{2,n}, \alpha_n)\middle|(X_{1,n}, X_{2,n}, \alpha_n) = (x_1, x_2, i)\right] = \rho\sigma_1(i)\sigma_2(i)h. \tag{3.16}$$

Lemma 2. *Under Assumption 1, the interpolation process $\{(X_1^h(t), X_2^h(t)) : 0 \le t \le T\}$ is tight in $D([0, T] : \mathbb{R}^2)$, the space of functions that are right continuous and have left limits, endowed with the Skorohod topology.*

Proof. It is done using a similar argument as in [30, Lemma 5.6]. □

Note that it can be shown by applying [39, Theorem 3.1] that the interpolated process $\{\alpha^h(\cdot)\}$ defined by (3.1)–(3.4) is tight. It then follows that $\{X_1^h(\cdot), X_2^h(\cdot), \alpha^h(\cdot)\}$ is tight in $D([0, T]; \mathbb{R}^2 \times \mathcal{M})$, the space of functions that are defined on $[0, T]$ taking values in $\mathbb{R}^2 \times \mathcal{M}$, and that are right continuous and have left limits, endowed with the Skorohod topology.

Since $\{X_1^h(\cdot), X_2^h(\cdot), \alpha^h(\cdot)\}$ is tight, by Prohorov's Theorem, it has convergent subsequences. We select such a convergent subsequence and still denote it by $\{X_1^h(\cdot), X_2^h(\cdot), \alpha^h(\cdot)\}$ for notational simplicity. Denote the limit by $\{X_1(\cdot), X_2(\cdot), \alpha(\cdot)\}$. By Skorohod representation, we may assume without loss of generality that the subsequence $\{X_1^h(\cdot), X_2^h(\cdot), \alpha^h(\cdot)\}$ converges to $\{X_1(\cdot), X_2(\cdot), \alpha(\cdot)\}$ w.p.1, and that the convergence is uniform on any bounded interval. The following theorem characterizes the limit process $\{X_1(\cdot), X_2(\cdot), \alpha(\cdot)\}$ via a martingale problem formulation method and establishes the weak convergence of the process $\{X_1^h(\cdot), X_2^h(\cdot), \alpha^h(\cdot)\}$.

Theorem 1. *Under Assumption 1, the interpolation process $\{X_1^h(\cdot), X_2^h(\cdot), \alpha^h(\cdot)\}$ converges weakly to $\{X_1(\cdot), X_2(\cdot), \alpha(\cdot)\}$ as $h \to 0$ such that $\{X_1(\cdot), X_2(\cdot), \alpha(\cdot)\}$ is the solution of the martingale problem associated with the operator $\mathcal{L}$, where $\mathcal{L}$ is defined* *as follows: for each $i \in \mathcal{M}$ and for any twice continuously differentiable function $\varphi(\cdot,\cdot,i)$,*

$$\mathcal{L}\varphi(x_1,x_2,i) = a_1(i)\frac{\partial\varphi}{\partial x_1}(x_1,x_2,i) + a_2(i)\frac{\partial\varphi}{\partial x_2}(x_1,x_2,i) + \frac{1}{2}\sigma_1^2(i)\frac{\partial^2\varphi}{\partial x_1^2}(x_1,x_2,i) + \frac{1}{2}\sigma_2^2(i)\frac{\partial^2\varphi}{\partial x_2^2}(x_1,x_2,i) + \rho\sigma_1(i)\sigma_2(i)\frac{\partial^2\varphi}{\partial x_1\partial x_2}(x_1,x_2,i) + \sum_{j=1}^{m_0} q_{ij}\varphi(x_1,x_2,j), \tag{3.17}$$

where $Q = (q_{ij})_{m_0\times m_0}$ is the generator of α_t.

Proof. It can be done following the five steps in the proof of [30, Theorem 5.8]. We refer the reader to [30] for details. □

An immediate consequence of the weak convergence result is the convergence of the European option prices, stated in the following corollary.

Corollary 1. *Let $V(x_1,x_2,i)$ denote the exact price of an European option written on the two correlated assets using the continuous-time regime-switching models, and let $V^h(x_1,x_2,i)$ denote the approximated price calculated from the lattice using step size h. Then we have*

$$\lim_{h\to 0} V^h(x_1,x_2,i) = V(x_1,x_2,i), \quad x_1\in\mathbb{R}, x_2\in\mathbb{R}, i\in\mathcal{M}.$$

Remark 2. While the convergence of European option prices is implied by the weak convergence of the discrete tree approximations of the underlying processes, the convergence of American option prices is much more complicated due to the early exercise feature of American options (determining the optimal exercise boundary is part of the solution). We have not proven the convergence of American option prices in the regime-switching models in the present paper and leave it for future study.

4. Numerical study

In this section we provide two numerical examples of using the lattice approach to price both European and American options on the maximum and minimum of two assets in the regime-switching model. For the European options, we implement a Monte-Carlo simulation approach and numerically compare it with our lattice.

4.1. A Monte-Carlo simulation approach for European options on the maximum and minimum of two assets in the regime-switching model

Analytical formulas are available (see Stulz [40]) for European call and put options on the maximum and minimum of two correlated assets, whose prices follow the geometric Brownian motion (GBM) without regime-switching. In this section, we combine the Monte-Carlo simulation approach with the analytical solutions to obtain an algorithm for pricing European options on the maximum and minimum of two correlated assets whose prices are governed by the regime-switching model (2.1). To proceed, let $\mathcal{F}_T$ be the σ-algebra generated by the Markov chain α_t, $0\le t\le T$, i.e., $\mathcal{F}_T = \sigma\{\alpha_t, 0\le t\le T\}$. Define the sample interest rate, and the sample volatility rates for the two assets by

$$R = \frac{1}{T}\int_0^T r(\alpha_t)dt, \qquad V_1 = \left(\frac{1}{T}\int_0^T \sigma_1^2(\alpha_t)dt\right)^{\frac{1}{2}}, \qquad V_2 = \left(\frac{1}{T}\int_0^T \sigma_2^2(\alpha_t)dt\right)^{\frac{1}{2}}. \tag{4.1}$$

For a given realization of the Markov chain $\{\alpha_t : 0\le t\le T\}$, the corresponding option price can be calculated by the analytical formula. For instance, consider an European call option on the minimum of the two assets with maturity T and exercise price K. Then the option price at time zero, denoted by $C_0^{\min}$, is given by

$$\begin{aligned} C_0^{\min} &= E\left\{\exp\left(-\int_0^T r(\alpha_t)dt\right)(\min(S_1(T),S_2(T))-K)^+\right\} \\ &= E\left\{E\left[e^{-RT}(\min(S_1(T),S_2(T))-K)^+\middle|\mathcal{F}_T\right]\right\}, \end{aligned} \tag{4.2}$$

where the sample path option price is given by the analytical formula [40]:

$$\begin{aligned} E\left[e^{-RT}(\min(S_1(T),S_2(T))-K)^+\middle|\mathcal{F}_T\right] &= S_1(0)N_2\left(\gamma_1 + V_1\sqrt{T}, \frac{\ln\left(\frac{S_2(0)}{S_1(0)}\right) - \frac{1}{2}V^2T}{V\sqrt{T}}, \frac{\rho V_2 - V_1}{V}\right) \\ &+ S_2(0)N_2\left(\gamma_2 + V_2\sqrt{T}, \frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right) - \frac{1}{2}V^2T}{V\sqrt{T}}, \frac{\rho V_1 - V_2}{V}\right) - Ke^{-RT}N_2(\gamma_1,\gamma_2,\rho), \end{aligned} \tag{4.3}$$

where $N_2(\alpha, \beta, \theta)$ is the bivariate cumulative standard normal distribution with upper limits of integration α and β and correlation coefficient θ, and

$$\gamma_1 = \frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right) + (R - \frac{1}{2}V_1^2)T}{V_1\sqrt{T}}, \qquad \gamma_2 = \frac{\ln\left(\frac{S_2(0)}{S_1(0)}\right) + (R - \frac{1}{2}V_2^2)T}{V_2\sqrt{T}},$$

$$V = \left(V_1^2 + V_2^2 - 2\rho V_1 V_2\right)^{\frac{1}{2}}.$$

The implementation of the Monte-Carlo simulation approach is stated in the following algorithm.

Algorithm:

Let N be the number of replications.

For $n = 1, \ldots, N$,

1. obtain the nth sample path of α_t, $0 \le t \le T$;
2. calculate V_1, V_2 and R associated with the sample path;
3. calculate the analytical price $C_{0,n}^{\min}$ for the nth sample path, using (4.3);
4. calculate the average $C_0^{\min} \approx \frac{1}{N}\sum_{n=1}^{N} C_{0,n}^{\min}$.

For random sampling of Markov chains, we follow the method in Yin and Zhang [41] (Section 4, Chapter 2).

4.2. Numerical results

Example 1 (*Two Regimes*)**.** We consider a model with two regimes, namely $m_0 = 2$. In this case, the generator of the Markov chain α_t takes the form of

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \quad q_{12} > 0, q_{21} > 0,$$

where q_{12} is the jump rate from regime 1 to regime 2 and q_{21} is the jump rate from regime 2 to regime 1.

The model parameters are specified as follows: $q_{12} = 6, q_{21} = 9, \sigma_1(1) = 0.35, \sigma_1(2) = 0.2, \sigma_2(1) = 0.25, \sigma_2(2) = 0.15$, $r(1) = r(2) = 0.05$, $\rho = 0.5$. All options have maturity $T = 1$. The initial asset prices are $S_1(0) = 30$, $S_2(0) = 30$. The exercise price of option is $K = 25, 30, 35$.

For the lattice implementation, we choose $\bar{\sigma}_1 = 0.25$, $\bar{\sigma}_2 = 0.2$, $l_{1,1} = 2$, $l_{1,2} = 1$, $l_{2,1} = 2$, $l_{2,2} = 1$. Those values satisfy Assumption 1.

Table 1 reports the results of the time-zero prices for three types of options: European call options on the maximum of the two assets, European put options on the minimum of the two assets, and American put options on the minimum of the two assets. Three different numbers for the time steps ($N = 50, 80, 100$) are used to illustrate the convergence of the lattice approximation. For the European options, results obtained using the Monte-Carlo simulation algorithm are reported in the last column in Table 1, for which the first number is the estimated option price and the second one (within parentheses) is the standard deviation which is computed based on 10 approximations, each being done using 30 000 sample paths of the Markov chain α_t, $0 \le t \le T$.

In Table 1, we observe the convergence trend of the approximated European option prices as the number of time steps increases (if the option prices obtained by the Monte-Carlo simulations are considered as "benchmark" for true values). We also observe that the American option price is always greater than its European counterpart. The positive difference is the early exercise premium, which increases as the exercise price increases for the put options.

To further explore the impact of regime-switching on the option prices, we calculate the at-the-money European and American put options on the minimum of two assets for different values of the jump rate q_{12}, that is, the same model parameters are used (specifically, for the at-the-money options, the exercise price is set to $K = 30$) except for the jump rate q_{12}, which increases from $q_{12} = 3$ to $q_{12} = 9$. The results are reported in Table 2. Note that q_{12} is the jump rate from regime 1 to regime 2, and a large value of q_{12} means a small chance for the market to stay in regime 1. Also note that both assets have large volatility values in regime 1 which implies large option values. This is observed in Table 2 as both European and American values decrease as q_{12} increases.

Example 2 (*Four Regimes*)**.** In this example we consider a four-state Markov chain α_t with state space $\mathcal{M} = \{1, 2, 3, 4\}$ and generator

$$Q = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

Thus the market can be in any of the four regimes with equal probability. The model parameters are chosen as follows. For asset $S_1(t)$, $\sigma_1(1) = 0.55$, $\sigma_1(2) = 0.45$, $\sigma_1(3) = 0.35$, $\sigma_1(4) = 0.25$. For asset $S_2(t)$, $\sigma_2(1) = 0.35$, $\sigma_2(2) = 0.25$, $\sigma_2(3) =$

Table 1
European and American options on two assets (two regimes).

Exercise price	Number of steps			MC simulations
	50	80	100	
	European call on maximum (Regime 1)			
25	9.6476	9.6493	9.6498	9.6510 (0.005)
30	5.7924	5.7958	5.7969	5.8027 (0.004)
35	3.1169	3.1177	3.1183	3.1231 (0.004)
	European call on maximum (Regime 2)			
25	9.5198	9.5252	9.5269	9.5342 (0.003)
30	5.6353	5.6432	5.6458	5.6548 (0.002)
35	2.9596	2.9649	2.9669	2.9766 (0.004)
	European put on minimum (Regime 1)			
25	1.2293	1.2311	1.2312	1.2322 (0.003)
30	3.5707	3.5738	3.5748	3.5777 (0.005)
35	7.1624	7.1633	7.1640	7.1653 (0.005)
	European put on minimum (Regime 2)			
25	1.1346	1.1391	1.1400	1.1456 (0.003)
30	3.4305	3.4376	3.4400	3.4496 (0.004)
35	7.0205	7.0255	7.0275	7.0356 (0.004)
	American put on minimum (Regime 1)			
25	1.2731	1.2751	1.2752	
30	3.7194	3.7222	3.7232	
35	7.4737	7.4750	7.4757	
	American put on minimum (Regime 2)			
25	1.1725	1.1775	1.1785	
30	3.5651	3.5724	3.5748	
35	7.3067	7.3129	7.3152	

Table 2
Sensitivity of European and American puts on the minimum of two assets with respect to the jump rate q_{12} (two regimes).

Regime	Jump rate q_{12}						
	3	4	5	6	7	8	9
	European put on minimum						
1	3.8291	3.7309	3.6455	3.5707	3.5046	3.4458	3.3931
2	3.6643	3.5753	3.4981	3.4305	3.3707	3.3176	3.2699
	American put on minimum						
1	3.9751	3.8779	3.7935	3.7194	3.6540	3.5958	3.5436
2	3.7951	3.7075	3.6316	3.5651	3.5063	3.4540	3.4072

0.15, $\sigma_2(4) = 0.10$. For the interest rates, $r(1) = r(2) = r(3) = r(4) = 0.05$. All options have maturity $T = 1$. The initial asset prices are $S_1(0) = 100$, $S_2(0) = 100$. The exercise price of option is $K = 90, 100, 110$. For the lattice implementation, we choose $\bar{\sigma}_1 = 0.30, \bar{\sigma}_2 = 0.20, l_{1,1} = 2, l_{1,2} = 2, l_{1,3} = 2, l_{1,4} = 1, l_{2,1} = 2, l_{2,2} = 2. l_{2,3} = 1, l_{2,4} = 1$. Those hypothesized values satisfy Assumption 1. In Tables 3 and 4 we report and compare the results obtained from the lattice approach and from the Monte-Carlo simulations for two types of European options. In Table 5 we report the approximation values from the lattice approach for the American put options on the minimum of the two assets.

Tables 3–5 show again the convergence trend of the approximated European option prices and the early exercise premiums for American option prices. Moreover, it is interesting to note that while all options reported in Table 1 (two-regime model) are increasing as the number of time steps increases, this is not the case for the four-regime model in Tables 3–5 from which we observe both increasing and decreasing option values. It will be an interesting and challenging future research topic to establish the error bounds and rates of convergence for the lattice constructed in this paper for regime-switching models.

We report the time used to complete an entire tree computation that produces the approximated option prices at all tree nodes and for every regime. The computations were performed using MATLAB on a notebook PC with the following system specifications: Intel(R) Core(TM) i5 CPU M540 2.53 GHz 4 GB RAM. For the two-regime example, it took 35.4 s ($n = 50$), 145.8 s ($n = 80$), 280.1 s ($n = 100$) for European options, and 37.6 s ($n = 50$), 148.0 s ($n = 80$), 292.9 s ($n = 100$) for American options. For the four-regime example, it took 104.9 s ($n = 50$), 433.4 s ($n = 80$), 851.5 s ($n = 100$) for European options, and 107.9 s ($n = 50$), 442.8 s ($n = 80$), 862.4 s ($n = 100$) for American options.

We note that it is also very convenient to use the lattice constructed in this paper to calculate the approximated delta (Δ) and gamma (Γ) values with respect to each underlying asset and for each regime. The way is very similar to the CRR binomial

Table 3
European call on the maximum of the two assets (four regimes).

Exercise price	Regime	Number of steps			MC simulations
		50	80	100	
90	1	32.6571	32.6280	32.6175	32.5372 (0.017)
	2	31.0877	31.0780	31.0738	31.0262 (0.013)
	3	29.8086	29.8135	29.8145	29.7819 (0.014)
	4	28.9299	28.9415	28.9471	28.9275 (0.018)
100	1	25.7567	25.7285	25.7191	25.6599 (0.022)
	2	24.0138	24.0083	24.0064	23.9866 (0.018)
	3	22.5537	22.5652	22.5690	22.5648 (0.017)
	4	21.5751	21.5939	21.6001	21.6061 (0.023)
110	1	20.1029	20.0681	20.0546	19.9803 (0.019)
	2	18.3023	18.2907	18.2867	18.2451 (0.015)
	3	16.7873	16.7954	16.7959	16.7754 (0.017)
	4	15.7768	15.7858	15.7906	15.7886 (0.021)

Table 4
European put on the minimum of the two assets (four regimes).

Exercise price	Regime	Number of steps			MC simulations
		50	80	100	
90	1	11.7680	11.7386	11.7286	11.6809 (0.013)
	2	10.4756	10.4634	10.4590	10.4356 (0.015)
	3	9.3970	9.3966	9.3964	9.3934 (0.013)
	4	8.6425	8.6444	8.6516	8.6483 (0.013)
100	1	17.5920	17.5670	17.5588	17.5065 (0.017)
	2	16.1220	16.1174	16.1159	16.0981 (0.018)
	3	14.8581	14.8686	14.8721	14.8710 (0.021)
	4	13.9726	13.9925	13.9979	14.0053 (0.025)
110	1	24.5812	24.5517	24.5414	24.4698 (0.018)
	2	23.0608	23.0510	23.0473	23.0108 (0.018)
	3	21.7515	21.7585	21.7590	21.7385 (0.022)
	4	20.8362	20.8452	20.8493	20.8431 (0.026)

Table 5
American put on the minimum of the two assets (four regimes).

Exercise price	Regime	Number of steps		
		50	80	100
90	1	12.1396	12.1100	12.1001
	2	10.7847	10.7731	10.7690
	3	9.6556	9.6567	9.6570
	4	8.8641	8.8675	8.8747
100	1	18.1809	18.1530	18.1437
	2	16.6313	16.6248	16.6225
	3	15.2997	15.3091	15.3121
	4	14.3611	14.3774	14.3827
110	1	25.4170	25.3853	25.3743
	2	23.8018	23.7912	23.7872
	3	22.4091	22.4158	22.4165
	4	21.4192	21.4302	21.4349

tree for Black–Scholes–Merton's model [1]. Such results are crucial to the hedging of European and American options written on the two correlated assets in regime-switching models.

To conclude this section, we point out that the proposed lattice approach is particularly appealing for practical application of pricing and hedging American options written on two correlated assets, since it is not easy to implement the Monte-Carlo simulation approach for American options. One may resort to the finite-difference method for pricing American options. For the regime-switching models considered in this paper, the American option prices satisfy a system of m_0 free boundary value problems and an (optimal) early exercise boundary is associated with each regime. A numerical scheme using finite-difference can be developed to solve the system of free boundary value problems. We think that the proposed lattice approach is more convenient and intuitive than the finite-difference approach but certainly need to further study the relationship between the two approaches in the future.

5. Concluding remarks

We extend the regime-switching recombining tree approach for options on a single asset proposed in Liu [14] to options with two correlated underlying assets whose prices are governed by the regime-switching geometric Brownian motion (GBM) models. The new lattice approach achieves complete node recombination and grows at most quadratically as the number of time steps increases. The weak convergence of the discrete lattice approximations to the continuous-time regime-switching diffusion processes is established. The accuracy and convergence of the lattice method are illustrated by pricing both European and American options written on the maximum and minimum of two assets in different regimes.

References

[1] J. Cox, S. Ross, M. Rubinstein, Option pricing, a simplified approach, Journal of Financial Economics 7 (1979) 229–263.
[2] P. Boyle, A lattice framework for option pricing with two state variables, Journal of Financial and Quantitative Analysis 23 (1988) 1–12.
[3] P. Boyle, J. Evnine, S. Gibbs, Numerical evaluation of multivariate contingent claims, The Review of Financial Studies 2 (1989) 241–250.
[4] N. Ekvall, A lattice approach for pricing of multivariate contingent claims, European Journal of Operational Research 91 (1996) 214–228.
[5] D.D. Aingworth, S.R. Das, R. Motwani, A simple approach for pricing equlty options with Markov switching state variables, Quantitative Finance 6 (2006) 95–105.
[6] N.P.B. Bollen, Valuing options in regime-switching models, Journal of Derivatives 6 (1998) 38–49.
[7] P. Boyle, T. Draviam, Pricing exotic options under regime switching, Insurance: Mathematics & Economics 40 (2007) 267–282.
[8] J. Buffington, R.J. Elliott, American options with regime switching, International Journal of Theoretical and Applied Finance 5 (2002) 497–514.
[9] P. Eloe, R.H. Liu, J.Y. Sun, Double barrier option under regime-switching exponential mean-reverting process, International Journal of Computer Mathematics 86 (6) (2009) 964–981.
[10] X. Guo, Information and option pricings, Quantitative Finance 1 (2000) 38–44.
[11] X. Guo, Q. Zhang, Closed-form solutions for perpetual American put options with regime switching, SIAM Journal of Applied Mathematics 64 (2004) 2034–2049.
[12] M.R. Hardy, A regime-switching model for long-term stock returns, North American Actuarial Journal 5 (2001) 41–53.
[13] A.Q.M. Khaliq, R.H. Liu, New numerical scheme for pricing American option with regime-switching, International Journal of Theoretical and Applied Finance 12 (3) (2009) 319–340.
[14] R.H. Liu, Regime-switching recombining tree for option pricing, International Journal of Theoretical and Applied Finance 13 (3) (2010) 479–499.
[15] R.H. Liu, Q. Zhang, G. Yin, Option pricing in a regime switching model using the Fast Fourier Transform, Journal of Applied Mathematics and Stochastic Analysis 2006 (2006). Article ID 18109, http://dx.doi.org/10.1155/JAMSA/2006/18109.
[16] D.D. Yao, Q. Zhang, X.Y. Zhou, A regime-switching model for European options, in: H.M. Yan, G. Yin, Q. Zhang (Eds.), Stochastic Processes Optimization, and Control Theory: Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems, Springer, 2006, pp. 281–300.
[17] F.L. Yuen, H. Yang, Option pricing with regime switching by trinomial tree method, Journal of Computational and Applied Mathematics 233 (2010) 1821–1833.
[18] F.L. Yuen, H. Yang, Pricing Asian options and equity-indexed annuites with regime switching by the trinomial tree method, North American Actuarial Journal 14 (2) (2010) 256–277.
[19] R. Bansal, H. Zhou, Term structure of interest rates with regime shifts, Journal of Finance 57 (5) (2002) 1997–2043.
[20] C. Landén, Bond pricing in a hidden Markov model of the short rate, Finance and Stochastics 4 (2000) 371–389.
[21] R.H. Liu, A new tree method for pricing financial derivatives in a regime-switching mean-reverting model, Nonlinear Analysis: Real World Applications 13 (2012) 2609–2621.
[22] L. Clewlow, C. Strickland, Energy Derivatives: Pricing and Risk Management, Lacima Publications, London, UK, 2000.
[23] J. Lucia, E. Schwartz, Electricity prices and power derivatives: evidence from the Nordic power exchange, Review of Derivatives Research 5 (2002) 5–50.
[24] X. Zhou, G. Yin, Markowitz's mean-variance portfolio selection with regime switching: a continuous-time model, SIAM Journal on Control and Optimization 42 (2003) 1466–1482.
[25] P. Eloe, R.H. Liu, M. Yatsuki, G. Yin, Q. Zhang, Optimal selling rules in a regime-switching exponential Gaussian diffusion model, SIAM Journal on Applied Mathematics 69 (3) (2008) 810–829.
[26] G. Yin, R.H. Liu, Q. Zhang, Recursive algorithms for stock liquidation: A stochastic optimization approach, SIAM Journal on Control and Optimization 13 (2002) 240–263.
[27] G. Yin, Q. Zhang, F. Liu, R.H. Liu, Y. Cheng, Stock liquidation via stochastic approximation using NASDAQ daily and intra-day data, Mathematical Finance 16 (1) (2006) 217–236.
[28] Q. Zhang, Stock trading: An optimal selling rule, SIAM Journal on Control and Optimization 40 (2001) 64–87.
[29] Q. Zhang, G. Yin, R.H. Liu, A near-optimal selling rule for a two-time-scale market model, SIAM Journal of Multiscale Modeling & Simulation 4 (1) (2005) 172–193.
[30] G. Yin, C. Zhu, Hybrid Switching Diffusions: Properties and Applications, Springer, 2010.
[31] D.B. Nelson, K. Ramaswamy, Simple binomial processes as diffusion approximations in financial models, The Review of Financial Studies 3 (1990) 393–430.
[32] R.J. Elliott, L. Chan, T.K. Siu, Option pricing and Esscher transform under regime switching, Annals of Finance 1 (2005) 423–432.
[33] T.K. Siu, A game theoretic approach to option valuation under Markovian regime-switching models, Insurance: Mathematics & Economics 42 (3) (2008) 1146–1158.
[34] R.J. Elliott, T.K. Siu, A. Badescu, On pricing and hedging options in regime-switching models with feedback effect, Journal of Economic Dynamics and Control 35 (5) (2011) 694–713.
[35] T.K. Siu, Regime-switching risk: to price or not to price? International Journal of Stochastic Analysis 2011 (2011). Article ID 843246, http://dx.doi.org/10.1155/2011/843246.
[36] T.K. Siu, H. Yang, Option pricing when the regime-switching risk is priced, ACTA Mathematicae Applicatae Sinica (English Series) 25 (3) (2009) 369–388.
[37] A. Ben-Tal, A. Nemirovski, Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, SIAM, 2001.
[38] J. Nocedal, S. Wright, Numerical Optimization, second ed., Springer, 2006.
[39] G. Yin, Q. Zhang, G. Badowski, Discrete-time singularly perturbed Markov chains: Aggregation, occupation measures, and switching diffusion limit, Advances in Applied probability 35 (2003) 449–476.
[40] R.M. Stulz, Options on the minimum or the maximum of two risky assets, Journal of Financial Economics 10 (1982) 161–185.
[41] G. Yin, Q. Zhang, Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach, Springer, 1998.