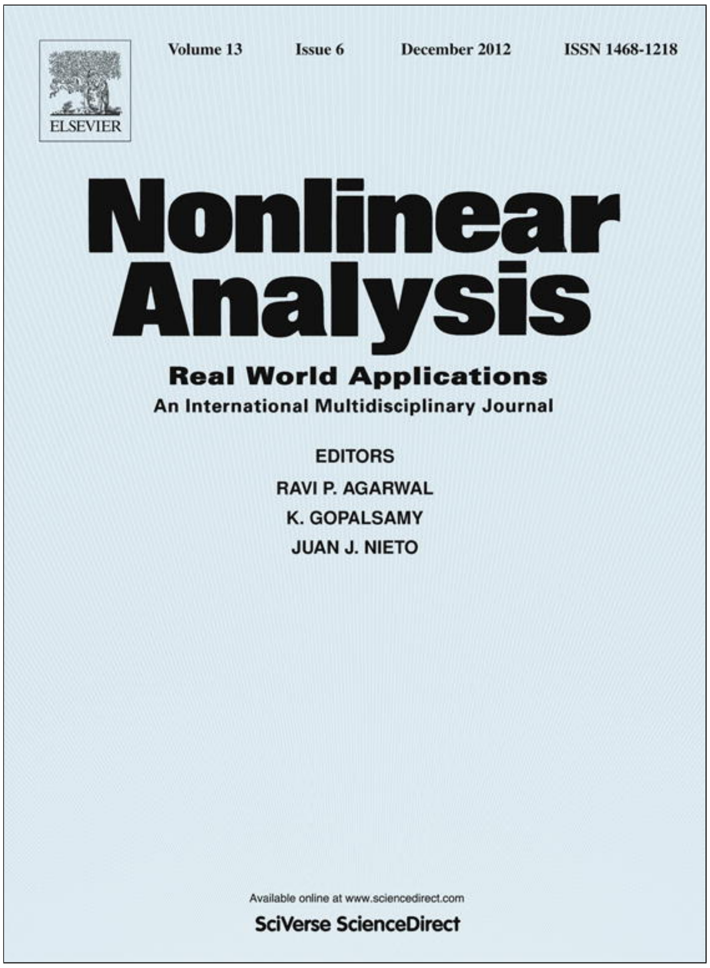
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# A new tree method for pricing financial derivatives in a regime-switching mean-reverting model

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ABSTRACT

This paper develops a new tree method for pricing financial derivatives in a regime-switching mean-reverting model. The tree achieves full node recombination and grows linearly as the number of time steps increases. Conditions for non-negative branch probabilities are presented. The weak convergence of the discrete tree approximations to the continuous regime-switching mean-reverting process is established. To illustrate the application in mathematical finance, the recombining tree is used to price commodity options and zero-coupon bonds. Numerical results are provided and compared.


## 1. Introduction

Regime-switching models are a class of stochastic nonlinear systems that exhibit both continuous dynamics and discrete events. They have drawn a significant amount of attention in recent years in a variety of applications due to their ability to model complex systems with uncertainty. Aiming to include the influence of macroeconomic factors on individual asset price dynamics, regime-switching models have been accepted in financial mathematics. In this setting, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents randomly changing economical factors. Model parameters (drift and volatility coefficients) are assumed to depend on the Markov chain and are allowed to take different values in different regimes. Regime-switching models have been used in various studies in financial mathematics including equity options [1–14], bond prices and interest rate derivatives [15–17], energy and commodity derivatives [18–20], portfolio selection[21], trading rules [22–26,5], and others. We refer the reader to Yin and Zhu [27] on the recent development of regime-switching systems and applications.

Binomial trees have provided a useful numerical technique in computational finance and have been widely used in the finance and insurance industries. The ample literature exists. See, e.g., [28,2,29–35,11,36–38] and the references therein. In principle binomial trees are a class of discrete approximation methods to the continuous-time stochastic systems. Nelson and Ramaswamy [36] call a binomial tree approximation of a continuous diffusion "computationally simple" if the number of nodes grows at most linearly in the number of time intervals. Thus an important consideration in designing a tree algorithm is to make sure that the number of nodes do not grow too fast as the number of time steps increases. In many cases, the goal is to have a computationally simple tree.

A number of articles in the literature are concerned with the development of discrete tree methods for regime-switching diffusions. See, e.g., [1,2,7,11]. Guo [7], and Aingworth et al. [1] study equity option prices assuming that the underlying equity price follows a regime-switching geometric Brownian motion (GBM) model. They both extend the CRR binomial tree (Cox, Ross and Rubinstein [30]) in a straightforward manner, resulting in a partially recombining tree for the regime-switching GBM model. However, as noted in Liu [11], this tree is not computationally simple; namely, the number of nodes

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grows faster than a linear function of the number of time steps. Bollen [2] considers a two-regime GBM model and proposes a new pentanomial tree that achieves complete node recombination. Consequently, his pentanomial tree does grow linearly. Liu [11] adapts a similar idea and develops a linear tree for the regime-switching GBM model with any number of regimes. It was seen in [11] that the new recombining tree is efficient in pricing both European and American equity options.

While the GBM model has been well accepted for equity prices, mean-reverting diffusion processes have been frequently used for stochastic interest rates, energy and commodity prices. See, e.g., [39,19,40,41] and the references therein. Hull and White [42,43] propose a trinomial tree for a mean-reverting interest rate model (the Hull–White model) and use it to price bond prices and other interest rate derivatives. They also note that for interest rates, a trinomial tree (with up and down boundaries due to the mean-reverting feature) is usually better than a binomial approximation. Similar trees are presented for other interest rate models. See [39,36] for details.

When dealing with regime-switching mean-reverting models, we encounter the same difficulty as the one with the regime-switching GBM model aforementioned, that is, a direct extension of the trinomial tree will not produce a fully recombining tree. Special care needs to be taken in order to have a linear tree. In this work we propose a recombining tree for the regime-switching mean-reverting model defined in (2.1). Our design achieves full node recombination and grows linearly as the number of time steps increases. Moreover, we present conditions under which the branch probabilities are non-negative. We establish the weak convergence of the discrete tree approximations to the continuous process. To illustrate the application in mathematical finance, we provide two examples: (I) pricing commodity options assuming a regime-switching exponential mean-reverting process for the commodity prices, (II) pricing zero-coupon bonds assuming a regime-switching mean-reverting model for the interest rates.

The paper is organized as follows. Section 2 is devoted to the tree development. The regime-switching mean-reverting model is introduced first. The construction of the new recombining tree is presented next. Moreover, Conditions for non-negative branch probabilities are presented. The proof is deferred to the appendix. Section 3 is concerned with the weak convergence of the tree approximations to the continuous process. Section 4 presents two application examples. We use the tree to price commodity options assuming a regime-switching exponential mean-reverting process for the commodity prices. We use the tree to price zero-coupon bonds assuming a regime-switching mean-reverting model for the interest rates. Numerical results are provided and compared. Section 5 provides further remarks and concludes the paper.

## 2. Regime-switching mean-reverting model and recombining tree

Let the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ be fixed, upon which all stochastic processes are defined. Let $\alpha_t$ be a continuous-time Markov chain valued in $\mathcal{M} := \{1, \ldots, m_0\}$, a finite state space. The states of $\alpha_t$ represent general market trends and other economic factors (also known as state of the world or regime) and are labeled by integers 1 to $m_0$, where $m_0$ is the total number of states considered for the economy. For example, with $m_0 = 2$, $\alpha_t = 1$ may stand for an up market (a bullish market) and $\alpha_t = 2$ a down market (a bearish market). Let $B_t$ be a real-valued standard Brownian motion. Assume $\alpha_t$ is independent of $B_t$.

In this paper we consider the following regime-switching mean-reverting diffusion:

$$dx_t = b_{\alpha_t}(a_{\alpha_t} - x_t)dt + \sigma_{\alpha_t} dB_t, \tag{2.1}$$

where $x_t$ represents an underlying variable (for example, the stochastic interest rate), $a_{\alpha_t}$ denotes the mean reverting level, $b_{\alpha_t}$ denotes the rate at which $x_t$ is pulled back to the level $a_{\alpha_t}$, $\sigma_{\alpha_t}$ is the volatility. Note that the parameters $a$, $b$, and $\sigma$ in (2.1) depend on $\alpha_t$, indicating that they can take different values for different regimes. We assume that $b_i > 0$ and $\sigma_i > 0$ for $i = 1, \ldots, m_0$. We also assume that the initial value $x_0$ is a given constant.

We next present the details of our tree construction. Let $h > 0$ and $\bar{\sigma} > 0$ be the time step size and space step size, respectively. Consider the joint Markov process $(x_t, \alpha_t)$, $t \geq 0$. Let $(x_k, \alpha_k) := (x_t, \alpha_t)_{t=kh}$ be the state of $(x_t, \alpha_t)$ at the $k$th step of the tree, $k = 0, 1, 2, \ldots$. A value for the state $(x_k, \alpha_k)$ is specified for each node in the tree. Consider the node with $(x_k, \alpha_k) = (x, i)$. We now show how this state evolves in the next step. First, the Markov chain $\alpha_{k+1}$ may stay at the state $i$ with probability $p^{\alpha}_{ii}$ or jump to any other state $j \neq i$ with probability $p^{\alpha}_{ij}$, where the one-step transition probabilities $p^{\alpha}_{ij}$ of the Markov chain $\alpha_k$ are defined by

$$p^{\alpha}_{ij} = P\{\alpha_{k+1} = j | \alpha_k = i\}, \quad 1 \leq i, j \leq m_0. \tag{2.2}$$

Next, $x_{k+1}$ takes three values, resulting in three branches emanating from the current node. We call them top, middle and bottom branches, respectively. Depending on the value $x_k = x$, one of the three different structures is chosen. Specifically, let the integer $l_i$ be the number of upward (downward) moves, let $x^{i,1}$ (resp. $x^{i,2}$) be the lower bound (resp. upper bound) for $x_k$. The parameters $l_i$, $x^{i,1}$, $x^{i,2}$ will be discussed at the end of this section in Proposition 1. Then, we have the following three cases for $x_{k+1}$.

**Case 1.** If $x^{i,1} \leq x \leq x^{i,2}$, then the three branches for $x_{k+1}$ are given by $x + l_i\bar{\sigma}\sqrt{h}$ (top), $x$ (middle), and $x - l_i\bar{\sigma}\sqrt{h}$ (bottom). See Fig. 1 (Case 1, left). Let $p_{i,t}$, $p_{i,m}$ and $p_{i,b}$ denote the conditional probabilities corresponding to the top, middle and bottom

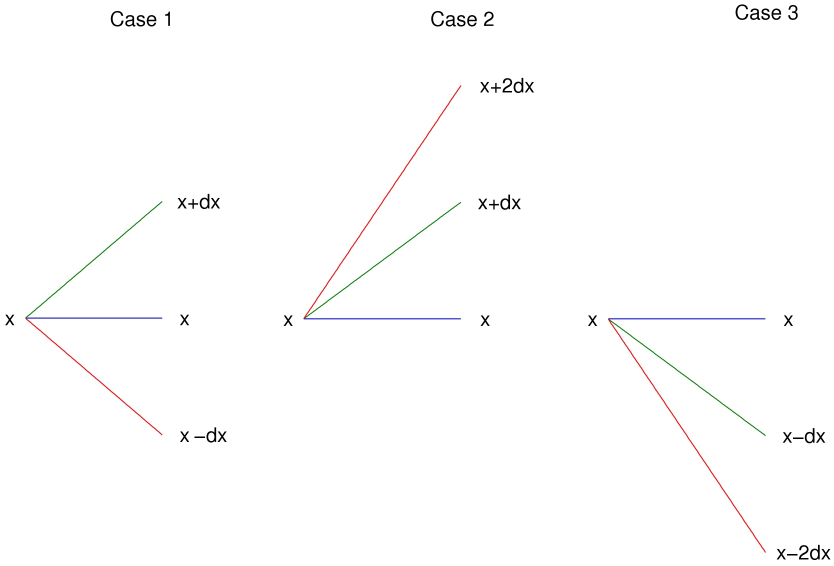


**Fig. 1.** Trinomial structures used in the regime-switching tree ($dx = l_i\bar{\sigma}\sqrt{h}$).

branches, respectively. By matching the mean and variance implied by this trinomial lattice to that implied by the diffusion (2.1), we have,

$$\begin{cases} (l_i\bar{\sigma}\sqrt{h})p_{i,t} - (l_i\bar{\sigma}\sqrt{h})p_{i,b} = \lambda_i(x)h, \\ (l_i\bar{\sigma}\sqrt{h})^2 p_{i,t} + (l_i\bar{\sigma}\sqrt{h})^2 p_{i,b} = \sigma_i^2 h + \lambda_i^2(x)h^2, \\ p_{i,t} + p_{i,m} + p_{i,b} = 1, \end{cases} \tag{2.3}$$

where

$$\lambda_i(x) = b_i(a_i - x). \tag{2.4}$$

It follows that,

$$\begin{aligned} p_{i,t} &= \frac{\sigma_i^2 + \lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{2(l_i\bar{\sigma})^2}, \\ p_{i,b} &= \frac{\sigma_i^2 - \lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{2(l_i\bar{\sigma})^2}, \\ p_{i,m} &= 1 - \frac{\sigma_i^2 + \lambda_i^2(x)h}{(l_i\bar{\sigma})^2}. \end{aligned} \tag{2.5}$$

**Case 2.** If $x < x^{i,1}$, then the three branches for $x_{k+1}$ are given by $x + 2l_i\bar{\sigma}\sqrt{h}$ (top), $x + l_i\bar{\sigma}\sqrt{h}$ (middle), and $x$ (bottom). See Fig. 1 (Case 2, middle). The equations satisfied by the conditional probabilities $p_{i,t}$, $p_{i,m}$ and $p_{i,b}$ are

$$\begin{cases} 2(l_i\bar{\sigma}\sqrt{h})p_{i,t} + (l_i\bar{\sigma}\sqrt{h})p_{i,m} = \lambda_i(x)h, \\ 4(l_i\bar{\sigma}\sqrt{h})^2 p_{i,t} + (l_i\bar{\sigma}\sqrt{h})^2 p_{i,m} = \sigma_i^2 h + \lambda_i^2(x)h^2, \\ p_{i,t} + p_{i,m} + p_{i,b} = 1. \end{cases} \tag{2.6}$$

which yield,

$$\begin{aligned} p_{i,t} &= \frac{\sigma_i^2 - \lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{2(l_i\bar{\sigma})^2}, \\ p_{i,m} &= -\frac{\sigma_i^2 - 2\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{(l_i\bar{\sigma})^2}, \\ p_{i,b} &= 1 + \frac{\sigma_i^2 - 3\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{2(l_i\bar{\sigma})^2}. \end{aligned} \tag{2.7}$$

**Case 3.** If $x > x^{i,2}$, then the three branches for $x_{k+1}$ are given by $x$ (top), $x - l_i\bar{\sigma}\sqrt{h}$ (middle), and $x - 2l_i\bar{\sigma}\sqrt{h}$ (bottom). See Fig. 1 (Case 3, right). The equations satisfied by $p_{i,t}$, $p_{i,m}$ and $p_{i,b}$ are

$$\begin{cases} (l_i\bar{\sigma}\sqrt{h})p_{i,m} + 2(l_i\bar{\sigma}\sqrt{h})p_{i,b} = -\lambda_i(x)h, \\ (l_i\bar{\sigma}\sqrt{h})^2 p_{i,m} + 4(l_i\bar{\sigma}\sqrt{h})^2 p_{i,b} = \sigma_i^2 h + \lambda_i^2(x)h^2, \\ p_{i,t} + p_{i,m} + p_{i,b} = 1, \end{cases} \tag{2.8}$$

In this case, we have

$$\begin{aligned} p_{i,m} &= -\frac{\sigma_i^2 + 2\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{(l_i\bar{\sigma})^2}, \\ p_{i,b} &= \frac{\sigma_i^2 + \lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{2(l_i\bar{\sigma})^2}, \\ p_{i,t} &= 1 + \frac{\sigma_i^2 + 3\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h}{2(l_i\bar{\sigma})^2}. \end{aligned} \tag{2.9}$$

Combining $x_{k+1}$ and $\alpha_{k+1}$, we have, emanating from the node $(x, i)$, there are $3m$ nodes for $(x_{k+1}, \alpha_{k+1})$, given by, respectively, for Case I,

$$(x_{k+1}, \alpha_{k+1}) = \begin{cases} (x + l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p_{ij}^{\alpha} p_{i,t}, \\ (x, j) & \text{with probability } p_{ij}^{\alpha} p_{i,m}, \quad j = 1, \ldots m_0, \\ (x - l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p_{ij}^{\alpha} p_{i,b}, \end{cases} \tag{2.10}$$

for Case II,

$$(x_{k+1}, \alpha_{k+1}) = \begin{cases} (x + 2l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p_{ij}^{\alpha} p_{i,t}, \\ (x + l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p_{ij}^{\alpha} p_{i,m}, \quad j = 1, \ldots m_0, \\ (x, j), & \text{with probability } p_{ij}^{\alpha} p_{i,b}. \end{cases} \tag{2.11}$$

and for Case III,

$$(x_{k+1}, \alpha_{k+1}) = \begin{cases} (x, j), & \text{with probability } p_{ij}^{\alpha} p_{i,t}, \\ (x - l_i\bar{\sigma}\sqrt{h}, j) & \text{with probability } p_{ij}^{\alpha} p_{i,m}, \quad j = 1, \ldots m_0, \\ (x - 2l_i\bar{\sigma}\sqrt{h}, j), & \text{with probability } p_{ij}^{\alpha} p_{i,b}, \end{cases} \tag{2.12}$$

**Remark 1.** Let $L = \max_{1\le i\le m_0} l_i$. Then $x_{k+1}$ always takes the three values from the set of $4L + 1$ values given by $\{x + j\bar{\sigma}\sqrt{h}, j = -2L, -2L+1, \ldots, 0, \ldots, 2L-1, 2L\}$, regardless of the regime $\alpha_k$. It can be seen that the number of nodes at the $k$th step of the tree is no more than $m_0(4Lk + 1)$, linear in $k$. Hence, our design does yield a computationally simple tree.

Note that if the parameters $h, \bar{\sigma}, l_i, x^{i,1}, x^{i,2}, i = 1, \ldots, m_0$, are not chosen properly, then (2.5) or (2.7), or (2.9) can produce unexpected negative branch probabilities. The following proposition presents a set of sufficient conditions under which the branch probabilities are ensured to be non-negative. Its proof is postponed to the appendix.

**Proposition 1.** *Choose $\bar{\sigma}, l_i, i = 1, \ldots, m_0$, such that*

$$\frac{2\sigma_i}{\sqrt{3}} \le l_i\bar{\sigma} \le 2\sigma_i, \quad i = 1, \ldots, m_0. \tag{2.13}$$

*Choose $h$ such that*

$$h \le \min_{1\le i\le m_0} \frac{2\sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{b_i l_i\bar{\sigma}}. \tag{2.14}$$

*Let*

$$x^{i,1} = a_i - \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{b_i\sqrt{h}}, \qquad x^{i,2} = a_i + \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{b_i\sqrt{h}}, \quad i = 1, \ldots, m_0. \tag{2.15}$$

*Then*

$$0 \le p_{i,t}, p_{i,m}, p_{i,b} \le 1, \quad i = 1, \ldots, m_0, \tag{2.16}$$

*where $p_{i,t}, p_{i,m}, p_{i,b}$ are given by (2.5) or (2.7), or (2.9).*

## 3. Weak convergence of the tree approximation

In this section we establish the weak convergence of the discrete approximation processes implied by the recombining tree to the continuous diffusion process (2.1). We apply the stochastic approximation methods developed in Yin and Zhu [27, Chapter 5] to our case. Since the proofs are similar, we only present the problem setup and the main results for weak convergence and refer the reader to [27] for technical details.

Fix $T > 0$. We consider the tree construction for the time period $[0, T]$. In what follows we will use $\varepsilon > 0$ (instead of $h$ as in Section 2) for the time step size. Then $N = \lfloor \frac{T}{\varepsilon} \rfloor$ is the number of steps of the tree,where $\lfloor z \rfloor$ denotes the floor of the real number $z$. Let the initial state $x_0$ and the initial regime $\alpha_0$ be given. For each fixed $\varepsilon > 0$, we construct a continuous-time process $(x^\varepsilon(t), \alpha^\varepsilon(t))$, $t \geq 0$ such that,

$$x^\varepsilon(0) = x_0, \qquad \alpha^\varepsilon(0) = \alpha_0, \tag{3.1}$$

$$x^\varepsilon(t) = x_n, \qquad \alpha^\varepsilon(t) = \alpha_n, \quad \text{for } n\varepsilon \leq t < (n+1)\varepsilon,\ n = 0, 1, \ldots, N-1, \tag{3.2}$$

where for given $(x_n, \alpha_n) = (x, i)$, $(x_{n+1}, \alpha_{n+1})$ is determined as following.

Let the generator $Q = (q_{ij})_{m_0 \times m_0}$ of the continuous-time Markov chain $\alpha_t$ be given. Then for the discrete Markov chain $\{\alpha_n, n \geq 0\}$, the probability transition matrix is approximated by

$$P^\varepsilon = I + \varepsilon Q, \tag{3.3}$$

that implies

$$P\{\alpha_{n+1} = j | \alpha_n = i\} = \delta_{ij} + \varepsilon q_{ij}, \quad 1 \leq i, j \leq m_0,\ 0 \leq n \leq N-1, \tag{3.4}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. It can be shown by applying [44, Theorem 3.1] that the interpolated process $\alpha^\varepsilon(\cdot)$ converges weakly to the continuous-time Markov chain $\alpha_t$ generated by $Q$.

Next, let $X^{\varepsilon,t}(x, i)$, $X^{\varepsilon,m}(x, i)$ and $X^{\varepsilon,b}(x, i)$ denote the three values for $x_{n+1}$ to take. Let $p^{\varepsilon,l}(x, i)$, $l = t, m, b$ denote the conditional probabilities for the top, middle, and bottom branches, respectively. Then,

$$P\{x_{n+1} = X^{\varepsilon,l}(x, i) | x_n = x, \alpha_n = i\} = p^{\varepsilon,l}(x, i), \quad l = t, m, b. \tag{3.5}$$

If the conditions of Proposition 1 are satisfied, then we have,

$$0 \leq p^{\varepsilon,l}(x, i) \leq 1, \quad l = t, m, b, \qquad \sum_{l=t,m,b} p^{\varepsilon,l}(x, i) = 1, \quad \text{for all } x \in \mathbb{R},\ i \in \mathcal{M}. \tag{3.6}$$

Moreover, Eqs. (2.3), (2.6) and (2.8) imply that, for $i \in \mathcal{M}$, $x \in \mathbb{R}$,

$$E[x_{n+1} | x_n = x, \alpha_n = i] = \sum_{l=t,m,b} p^{\varepsilon,l}(x, i) X^{\varepsilon,l}(x, i) = x + \lambda_i(x)\varepsilon, \tag{3.7}$$

and

$$\begin{aligned} E[(x_{n+1} - x)^2 | x_n = x, \alpha_n = i] &= \sum_{l=t,m,b} p^{\varepsilon,l}(x, i)(X^{\varepsilon,l}(x, i) - x)^2 \\ &= \sigma_i^2 \varepsilon + \lambda_i^2(x)\varepsilon^2. \end{aligned} \tag{3.8}$$

Using (3.7) and (3.8), we can establish in the following lemma an upper bound of the second moments of $x_n$, $0 \leq n \leq \lfloor \frac{T}{\varepsilon} \rfloor$.

**Lemma 1.** *Given $T > 0$. For any $\varepsilon > 0$ that satisfies (2.14), we have*

$$\sup_{0 \leq n \leq \lfloor T/\varepsilon \rfloor} E[x_n]^2 \leq (x_0^2 + CT)e^{CT} < \infty, \tag{3.9}$$

*where $C$ is a positive constant.*

Next, we establish the tightness of the continuous processes $\{x^\varepsilon(t) : 0 \leq t \leq T\}$ in a suitable function space. To this end, we rewrite the approximation (3.5) into the following iterative form:

$$x_{n+1} = x_n + b_{\alpha_n}(a_{\alpha_n} - x_n)\varepsilon + \xi(x_n, \alpha_n), \tag{3.10}$$

where the random functions $\xi(\cdot, \cdot)$ are defined as following. For each $x \in \mathbb{R}$ and each $i \in \mathcal{M}$, $\xi(x, i)$ takes the value $X^{\varepsilon,l}(x, i) - x - \lambda_i(x)\varepsilon$ (Recall that $\lambda_i(x) = b_i(a_i - x)$) with the probability $p^{\varepsilon,l}(x, i)$, $l = t, m, b$. It follows from (3.7) and (3.8) that

$$E[\xi(x_n, \alpha_n) | x_n = x, \alpha_n = i] = 0, \qquad E[\xi^2(x_n, \alpha_n) | x_n = x, \alpha_n = i] = \sigma_i^2 \varepsilon. \tag{3.11}$$

**Lemma 2.** *Assume that the conditions of Proposition 1 are satisfied. Then the interpolation process $\{x^\varepsilon(t) : 0 \le t \le T\}$ is tight in $D[0, T]$, the space of functions that are right continuous and have left limits, endowed with the Skorohod topology.*

Note that it can be shown by applying [44, Theorem 3.1] that the interpolated process $\{\alpha^\varepsilon(\cdot)\}$ is tight. It then follows that $\{x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ is tight in $D([0, T]; \mathbb{R} \times \mathcal{M})$, the space of functions that are defined on $[0, T]$ taking values in $\mathbb{R} \times \mathcal{M}$, and that are right continuous and have left limits, endowed with the Skorohod topology.

Since $\{x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ is tight, by Prohorov's Theorem, it has convergent subsequences. We select such a convergent subsequence and still denote it by $\{x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ for notational simplicity. Denote the limit by $\{x(\cdot), \alpha(\cdot)\}$. By Skorohod representation, we may assume without loss of generality that the subsequence $\{x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ converges to $\{x(\cdot), \alpha(\cdot)\}$ w.p.1, and that the convergence is uniform on any bounded interval. The following theorem characterizes the limit process $\{x(\cdot), \alpha(\cdot)\}$ via a Martingale problem formulation method and establishes the weak convergence of the process $\{x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$.

**Theorem 1.** *Assume that the conditions of Proposition 1 are satisfied. Then the interpolation process $\{x^\varepsilon(\cdot), \alpha^\varepsilon(\cdot)\}$ converges weakly to $\{x(\cdot), \alpha(\cdot)\}$ as $\varepsilon \to 0$ such that $\{x(\cdot), \alpha(\cdot)\}$ is the solution of the martingale problem associated with the operator $\mathcal{L}$, where $\mathcal{L}$ is defined as following: for each $i \in \mathcal{M}$ and for any twice continuously differentiable function $\varphi(\cdot, i)$,*

$$\mathcal{L}\varphi(x, i) = b_i(a_i - x)\frac{d\varphi}{dx}(x, i) + \frac{1}{2}\sigma_i^2\frac{d^2\varphi}{dx^2}(x, i) + \sum_{j=1}^{m_0} q_{ij}\varphi(x, j), \tag{3.12}$$

*where $Q = (q_{ij})_{m_0 \times m_0}$ is the generator of $\alpha_t$.*

**Proof.** It can be done following the five steps in the proof of [27, Theorem 5.8]. We refer the reader to [27] for details. □

## 4. Option and bond pricing using the recombining tree

In this section we provide two examples in mathematical finance for which we use the recombining tree. (I) We price commodity options assuming a regime-switching exponential mean-reverting process for the commodity prices. (II) We price zero-coupon bonds assuming a regime-switching mean-reverting model for the interest rates. We numerically compare the tree with other methods.

### 4.1. Commodity options

Mean-reversion is a distinguishing feature of commodity price changes and is well accepted in models for commodity prices. See, e.g., [19,6,41] and the references therein. We consider a certain commodity whose price $S_t$ at time $t \ge 0$ is given by $S_t = S_0 \exp(x_t)$, where $S_0 > 0$ is the initial price, and $x_t$ follows the regime-switching mean-reverting model (2.1) with $x_0 = 0$.

Using the recombining tree we have constructed in Section 2 for the underlying regime-switching process $x_t$, both European and American options written on the commodity $S_t$ can be priced following a similar procedure in Liu [11] used for valuing equity options. We start at the last time step of the tree ($n = N$) and work backward iteratively until we reach the time zero ($n = 0$). At each time step and for each tree node, we first calculate the probability weighted average of all option prices at these nodes in the next step that are directly emanated from the current node. We then discount the averaged future option value using the interest rate at the current node. This gives us the discounted expectation of the future option value, where the expectation is taken with respect to the risk-neutral probability. For American options, we need to in addition check the early exercise possibility by comparing this value with the immediate exercise value of the option. Specifically, let $V^n(x, i)$ denote a put option value at step $n$ for the node with state value $(x_n, \alpha_n) = (x, i)$. Then, at the last step $n = N$,

$$V^N(x, i) = \max\{K - S_0 e^x, 0\}, \quad i = 1, \ldots, m_0, \tag{4.1}$$

where $K$ is the strike price of the option. At step $0 \le n < N$, for European put,

$$V^n(x, i) = e^{-r_i h}\sum_{j=1}^{m_0} p_{ij}^\alpha \sum_{l=t,m,b} p^l(x, i)V^{n+1}(X^l(x, i), j), \tag{4.2}$$

where $X^l(x, i)$, $l = t, m, b$ denote the three values that $x_{n+1}$ takes, $p^l(x, i)$, $l = t, m, b$ denote the corresponding branch probabilities (see Section 2 for the three different cases), and $r_i$ is the risk-free interest rate at the node $(x, i)$ (we assume a regime-dependent risk-free interest rate $r_{\alpha_t}$.)

For American put option,

$$V^n(x, i) = \max\left\{K - S_0 e^x, e^{-r_i h}\sum_{j=1}^{m_0} p_{ij}^\alpha \sum_{l=t,m,b} p^l(x, i)V^{n+1}(X^l(x, i), j)\right\}. \tag{4.3}$$

Formulas for call options are obtained by simply changing $K - S_0 e^x$ to $S_0 e^x - K$ in (4.1)–(4.3).

**Table 1**
European put option prices in a two-regime model.

| $S_0$ | Tree $\alpha_0 = 1$ | PDE $\alpha_0 = 1$ | Tree $\alpha_0 = 2$ | PDE $\alpha_0 = 2$ |
|---|---|---|---|---|
| 94.0 | 6.2564 | 6.2548 | 5.7385 | 5.7376 |
| 96.0 | 5.2086 | 5.2065 | 4.8596 | 4.8587 |
| 98.0 | 4.2891 | 4.2869 | 4.0857 | 4.0850 |
| 100.0 | 3.4935 | 3.4919 | 3.4107 | 3.4105 |
| 102.0 | 2.8165 | 2.8143 | 2.8290 | 2.8280 |
| 104.0 | 2.2472 | 2.2449 | 2.3309 | 2.3295 |
| 106.0 | 1.7750 | 1.7728 | 1.9081 | 1.9067 |

**Table 2**
American put option prices in a two-regime model.

| $S_0$ | Tree $\alpha_0 = 1$ | PDE $\alpha_0 = 1$ | Tree $\alpha_0 = 2$ | PDE $\alpha_0 = 2$ |
|---|---|---|---|---|
| 94.0 | 8.3245 | 8.3164 | 9.2162 | 9.2015 |
| 96.0 | 7.0417 | 7.0331 | 7.9977 | 7.9831 |
| 98.0 | 5.8908 | 5.8824 | 6.8898 | 6.8753 |
| 100.0 | 4.8739 | 4.8660 | 5.8919 | 5.8781 |
| 102.0 | 3.9894 | 3.9816 | 5.0024 | 4.9894 |
| 104.0 | 3.2310 | 3.2236 | 4.2177 | 4.2053 |
| 106.0 | 2.5900 | 2.5833 | 3.5317 | 3.5200 |

**Example 1.** We consider a model with two regimes, i.e., $m_0 = 2$. In this case, the generator of the Markov chain $\alpha_t$ is given by

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \quad q_{12} > 0, \; q_{21} > 0,$$

where $q_{12}$ is the jump rate from regime 1 to regime 2 and $q_{21}$ is the jump rate from regime 2 to regime 1. We implement the regime-switching tree using the following parameter values: $q_{12} = q_{21} = 0.5$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $a_1 = 0.05$, $a_2 = 0.1$, $b_1 = 0.5$, $b_2 = 1.0$, $r_1 = 0.03$, $r_2 = 0.05$, the option maturity $T = 1$, the option strike price $K = 100$. We divide the life of option into 1000 steps. Thus the time step size $h = 1/1000$. We use $\bar{\sigma} = 0.1$ for the space step size.

For comparison, we employ an implicit finite difference method (see Hull, [40, Chapter 17]) to solve the pair of PDEs satisfied by the put option prices. Since $S_t = S_0 \exp(x_t)$, it follows from (2.1) that

$$dS_t = S_t \left( b_{\alpha_t}[a_{\alpha_t} - \ln(S_t/S_0)] + \frac{1}{2}\sigma_{\alpha_t}^2 \right) dt + S_t \sigma_{\alpha_t} dB_t. \tag{4.4}$$

The coupled PDEs are given as,

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_1^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + \left( b_1[a_1 - \ln(S/S_0)] + \dfrac{1}{2}\sigma_1^2 \right) S \dfrac{\partial V_1}{\partial S} - (r_1 + q_{12})V_1 + q_{12}V_2 = 0, \\ \dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma_2^2 S^2 \dfrac{\partial^2 V_2}{\partial S^2} + \left( b_2[a_2 - \ln(S/S_0)] + \dfrac{1}{2}\sigma_2^2 \right) S \dfrac{\partial V_2}{\partial S} - (r_2 + q_{21})V_2 + q_{21}V_1 = 0, \end{cases} \tag{4.5}$$

with terminal conditions $V_1(S, T) = V_2(S, T) = \max\{K - S, 0\}$, where $V_i(S, t)$ denotes the option price at time $0 \le t \le T$ when $S_t = S$ and $\alpha_t = i$, $i = 1, 2$.

We solve (4.5) on the rectangular region $[0, S_\infty] \times [0, T]$ for $(S, t)$ where $S_\infty = 200$ and $T = 1$. The region is discretized using grid sizes $\Delta S = 0.1$ and $\Delta t = 0.002$.

Tables 1 and 2 report the European put and American put option prices, respectively, at time $t = 0$ for seven different asset price $S_0$, using the two methods. The computations were performed using MATLAB on a notebook PC with the following system specifications: Intel(R) Core(TM)2 CPU T7400 2.16 GHz 1GB RAM. It takes around 43 s to finish the finite difference method and takes only about 4 s for the tree algorithm. We notice that the differences between the option prices are less than 0.5%.

### *4.2. Zero-coupon bonds*

We assume that the interest rate $r_t$ follows the regime-switching mean-reverting diffusion (2.1). That is,

$$dr_t = b_{\alpha_t}(a_{\alpha_t} - r_t)dt + \sigma_{\alpha_t} dB_t, \quad t \ge 0, \tag{4.6}$$

**Table 3**
Zero-coupon bond prices in a two-regime model.

| $T$ | Tree $\alpha_0 = 1$ | Landén $\alpha_0 = 1$ | Tree $\alpha_0 = 2$ | Landén $\alpha_0 = 2$ |
|---|---|---|---|---|
| 1 | 0.9311 | 0.9311 | 0.9352 | 0.9352 |
| 2 | 0.8699 | 0.8699 | 0.8769 | 0.8769 |
| 3 | 0.8150 | 0.8150 | 0.8231 | 0.8232 |
| 5 | 0.7183 | 0.7183 | 0.7266 | 0.7267 |
| 7 | 0.6343 | 0.6344 | 0.6241 | 0.6421 |
| 10 | 0.5270 | 0.5271 | 0.5335 | 0.5336 |
| 20 | 0.2844 | 0.2845 | 0.2879 | 0.2880 |
| 30 | 0.1535 | 0.1536 | 0.1554 | 0.1555 |

with deterministic initial rate $r_0 > 0$. Note that without the modulated Markov chain $\alpha_t$, (4.6) would become the well-known Vasicek model for short rate.

We consider a zero-coupon bond with maturity $T$ and face value 1. Let $P(t, r_t, \alpha_t, T)$ denote the bond value at time $0 \le t \le T$. Then,

$$P(t, r_t, \alpha_t, T) = E\left\{\exp\left(-\int_t^T r_s ds\right)\middle| \mathcal{F}_t\right\}, \tag{4.7}$$

where $\mathcal{F}_t$ is the $\sigma$-algebra generated by the Brownian motion $B_s$ and the Markov chain $\alpha_s$, $0 \le s \le t$. For notation simplicity, introducing

$$P_i := P(t, r, i, T) := P(t, r_t, \alpha_t, T)|_{r_t=r,\alpha_t=i}. \tag{4.8}$$

Then $P_i$, $i = 1, \dots, m_0$, satisfy the following system of $m_0$ partial differential equations,

$$\frac{\partial P_i}{\partial t} + b_i(a_i - r)\frac{\partial P_i}{\partial r} + \frac{1}{2}\sigma_i^2\frac{\partial^2 P_i}{\partial r^2} + \sum_{j\neq i} q_{ij}(P_j - P_i) = rP_i \tag{4.9}$$

with terminal conditions $P(T, r, i, T) = 1$, $i = 1, \dots, m_0$.

If $b_i = \kappa$, $i = 1, \dots, m_0$, where $\kappa > 0$ is a constant, then Landén [16] showed that the solution of (4.9) is given by

$$P_i = \exp[A_i(\tau) + B(\tau)r], \tag{4.10}$$

where $\tau = T - t$ is the time-to-maturity of the bond, the function $B(\tau)$ satisfies

$$\frac{dB(\tau)}{d\tau} + \kappa B(\tau) + 1 = 0, \qquad B(0) = 0, \tag{4.11}$$

and the functions $A_i(\tau)$, $i = 1, \dots, m_0$, satisfy a system of ordinary differential equations given by,

$$\frac{d\mathbf{U}(\tau)}{d\tau} = \left(Q + \operatorname{diag}\left\{\kappa a_i B(\tau) + \frac{1}{2}\sigma_i^2 B^2(\tau)\right\}\right)\mathbf{U}(\tau), \qquad \mathbf{U}(0) = \mathbb{1}_{m_0}, \tag{4.12}$$

where

$$\mathbf{U}(\tau) = (e^{A_1(\tau)}, e^{A_2(\tau)}, \dots, e^{A_{m_0}(\tau)})' \in \mathbb{R}^{m_0}, \tag{4.13}$$

and $\mathbb{1}_{m_0} = (1, \dots, 1)' \in \mathbb{R}^{m_0}$.

**Example 2.** In this example we compute the prices of the zero-coupon bonds by implementing the recombining tree of the model (4.6) for the interest rate $r_t$. We compare the results with that given by (4.10). The parameter values are $m_0 = 2$, $q_{12} = 3$, $q_{21} = 1$, $b_1 = b_2 = \kappa = 0.6$, $a_1 = 0.1$, $a_2 = 0.05$, $\sigma_1 = 0.03$, $\sigma_2 = 0.02$, the initial interest rate $r_0 = 0.07$. We use $h = 0.002$ and $\bar{\sigma} = 0.02$ for the time and space step sizes, respectively, in the tree implementation. Table 3 reports the zero-coupon bond prices at time $t = 0$ for eight different maturities $T$, using the two methods. We see that the numbers calculated using the tree method are accurate to at least 3 decimal places. We note that for this particular case, we use the analytical solution to validate the approximated values from the tree. In general, the analytical solution may not be available and the tree becomes an efficient approximation to the true solution.

## 5. Concluding remarks

Considerable attention has been drawn to regime-switching models recently in mathematical finance. Mean-reversion is a distinguishing feature of commodity prices and interest rates. In this paper we consider a regime-switching mean-reverting model and develop a new discrete tree to approximate the continuous model. The tree grows linearly as the number of time steps increases. Hence it is possible to use the tree with a large number of time steps to achieve high accuracy in computing derivative prices. We present two examples: pricing commodity options and zero-coupon bonds. Moreover, we present sufficient conditions for non-negative branch probabilities. We establish the weak convergence of the discrete tree approximations to the continuous regime-switching mean-reverting process. An interesting topic for future researches is to develop efficient trees for multiple correlated assets with regime-switching. It is also interesting to investigate the applications of the model in related studies, see for example, [45–47].

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## Appendix

**Proof of Proposition 1. Case 1.** $x^{i,1} \le x \le x^{i,2}$. We will show that

$$0 \le p_{i,t}, p_{i,m}, p_{i,b} \le 1, \tag{A.1}$$

where $p_{i,t}, p_{i,m}, p_{i,b}$ are given by (2.5). Note that it is readily seen that $0 \le p_{i,t}, p_{i,m}, p_{i,b} \le 1$ if $\lambda_i(x) = 0$, owing to the condition (2.13). Next we show that (A.1) holds when $\lambda_i(x) \neq 0$. a. $0 \le p_{i,t} \le 1$. First,

$$\sigma_i^2 + \lambda_i(x) l_i \bar{\sigma} \sqrt{h} + \lambda_i^2(x) h = \sigma_i^2 - \frac{1}{4}(l_i\bar{\sigma})^2 + \left(\lambda_i(x)\sqrt{h} + \frac{1}{2} l_i \bar{\sigma}\right)^2 \ge 0$$

due to the condition $l_i\bar{\sigma} \le 2\sigma_i$. It follows immediately that $p_{i,t} \ge 0$. For the proof of $p_{i,t} \le 1$, we introduce the function $f(z)$ defined by,

$$f(z) = \lambda_i^2(x) z^2 + \lambda_i(x) l_i \bar{\sigma} z + \sigma_i^2 - 2(l_i\bar{\sigma})^2 \tag{A.2}$$

and consider the following two subcases.

*Subcase* 1. $\lambda_i(x) > 0$. We have $f(z) \le 0$, if

$$0 \le z \le \frac{-l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{2\lambda_i(x)}.$$

Using (2.13), we can easily verify that

$$2\left[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}\right] \le -l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}, \tag{A.3}$$

which leads to

$$\sqrt{h} \le \frac{-l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{2[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}]/\sqrt{h}}.$$

In view of (2.4) and (2.15), we see that $x \ge x^{i,1}$ is equivalent to

$$\lambda_i(x) \le \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\sqrt{h}}.$$

It then follows that

$$\sqrt{h} \le \frac{-l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{2\lambda_i(x)}.$$

Hence $f(\sqrt{h}) \le 0$, implying that $p_{i,t} \le 1$.

*Subcase* 2. $\lambda_i(x) < 0$. We have $f(z) \le 0$, if

$$0 \le z \le \frac{l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{-2\lambda_i(x)}.$$

From (A.3) we have

$$2\left[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}\right] \le l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2},$$

which leads to

$$\sqrt{h} \le \frac{l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{2[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}]/\sqrt{h}}.$$

In view of (2.4) and (2.15), we see that $x \le x^{i,2}$ is equivalent to

$$-\lambda_i(x) \le \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\sqrt{h}}.$$

It then follows that

$$\sqrt{h} \le \frac{l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{-2\lambda_i(x)}.$$

Hence $f(\sqrt{h}) \le 0$, implying that $p_{i,t} \le 1$.

b. $0 \le p_{i,b} \le 1$. This can be shown by using the same arguments as in part a.

c. $0 \le p_{i,m} \le 1$. In view of (2.5), clearly $p_{i,m} \le 1$. Now we show that $p_{i,m} \ge 0$. Using (2.13), we can verify that

$$\left[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}\right]^2 \le (l_i\bar{\sigma})^2 - \sigma_i^2,$$

which leads to

$$h \le \frac{(l_i\bar{\sigma})^2 - \sigma_i^2}{\left[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}\right]^2/h}.$$

In view of (2.4) and (2.15), we see that $x^{i,1} \le x \le x^{i,2}$ implies

$$\lambda_i^2(x) \le \frac{\left[l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}\right]^2}{h}.$$

It follows that

$$h \le \frac{(l_i\bar{\sigma})^2 - \sigma_i^2}{\lambda_i^2(x)}.$$

Then we have

$$\frac{\lambda_i^2(x)h + \sigma_i^2}{(l_i\bar{\sigma})^2} \le 1,$$

implying that $p_{i,m} \ge 0$.

**Case 2.** $x < x^{i,1}$. In this case, $p_{i,t}, p_{i,m}, p_{i,b}$ are given by (2.7). Also we have

$$\lambda_i(x) > \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\sqrt{h}} > 0. \tag{A.4}$$

a. $0 \le p_{i,m} \le 1$. Let

$$g(z) = \lambda_i^2(x)z^2 - 2\lambda_i(x)l_i\bar{\sigma}z + \sigma_i^2. \tag{A.5}$$

Then $p_{i,m} = -\frac{g(\sqrt{h})}{(l_i\bar{\sigma})^2}$. Note that $g(z) \le 0$, if

$$\frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\lambda_i(x)} \le z \le \frac{l_i\bar{\sigma} + \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\lambda_i(x)}. \tag{A.6}$$

We now show that $z = \sqrt{h}$ satisfies (A.6). First, (A.4) is equivalent to

$$\sqrt{h} > \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\lambda_i(x)}.$$

Next, in view of our tree construction described in Section 2, we must have $x + l_i\bar{\sigma}\sqrt{h} \ge x^{i,1}$. (Otherwise, the node in the previous step associated with the state value $x + l_i\bar{\sigma}\sqrt{h}$ would take Case 2 structure and the value $x$ would never be reached.) It follows that

$$\lambda_i(x) \le \frac{l_i\bar{\sigma} - \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\sqrt{h}} + b_i l_i\bar{\sigma}\sqrt{h}. \tag{A.7}$$

In view of (2.14), we have

$$h \le \frac{2\sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{b_i l_i\bar{\sigma}},$$

which implies,

$$b_i l_i\bar{\sigma}\sqrt{h} \le \frac{2\sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\sqrt{h}}.$$

Using it in (A.7), we obtain,

$$\lambda_i(x) \le \frac{l_i\bar{\sigma} + \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\sqrt{h}},$$

which yields,

$$\sqrt{h} \le \frac{l_i\bar{\sigma} + \sqrt{(l_i\bar{\sigma})^2 - \sigma_i^2}}{\lambda_i(x)}. \tag{A.8}$$

It follows that $g(\sqrt{h}) \le 0$. Hence $p_{i,m} \ge 0$. On the other hand, note that

$$g(z) = [\lambda_i(x)z - l_i\bar{\sigma}]^2 + \sigma_i^2 - (l_i\bar{\sigma})^2 \ge \sigma_i^2 - (l_i\bar{\sigma})^2.$$

We have

$$p_{i,m} = -\frac{g(\sqrt{h})}{(l_i\bar{\sigma})^2} \le \frac{(l_i\bar{\sigma})^2 - \sigma_i^2}{(l_i\bar{\sigma})^2} < 1.$$

b. $0 \le p_{i,t} \le 1$. First, noting that

$$\sigma_i^2 - \lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \lambda_i^2(x)h = \sigma_i^2 - \frac{1}{4}(l_i\bar{\sigma})^2 + \left(\lambda_i(x)\sqrt{h} - \frac{1}{2}l_i\bar{\sigma}\right)^2 \ge 0$$

due to the condition $l_i\bar{\sigma} \le 2\sigma_i$. It follows immediately that $p_{i,t} \ge 0$. For the proof of $p_{i,t} \le 1$, we consider the function $\tilde{g}(z)$ defined by,

$$\tilde{g}(z) = \lambda_i^2(x)z^2 - \lambda_i(x)l_i\bar{\sigma}z + \sigma_i^2 - 2(l_i\bar{\sigma})^2. \tag{A.9}$$

We have $\tilde{g}(z) \le 0$, if

$$0 \le z \le \frac{l_i\bar{\sigma} + \sqrt{9(l_i\bar{\sigma})^2 - 4\sigma_i^2}}{2\lambda_i(x)}.$$

It is easy to verify that

$$\frac{l_i\bar{\sigma}+\sqrt{(l_i\bar{\sigma})^2-\sigma_i^2}}{\lambda_i(x)} \le \frac{l_i\bar{\sigma}+\sqrt{9(l_i\bar{\sigma})^2-4\sigma_i^2}}{2\lambda_i(x)}.$$

In view of (A.8) we have $\tilde{g}(\sqrt{h}) \le 0$. It then follows that $p_{i,t} \le 1$.

c. $0 \le p_{i,b} \le 1$. First,

$$\lambda_i^2(x)h - 3\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \sigma_i^2 \le \lambda_i^2(x)h - 2\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \sigma_i^2 = g(\sqrt{h}) \le 0,$$

in view of Case 2, a. Hence $p_{i,b} \le 1$. Next, we have,

$$\lambda_i^2(x)h - 3\lambda_i(x)l_i\bar{\sigma}\sqrt{h} + \sigma_i^2 + 2(l_i\bar{\sigma})^2 \ge \left(\lambda_i(x)\sqrt{h} - \frac{3}{2}l_i\bar{\sigma}\right)^2 + \sigma_i^2 - \frac{1}{4}(l_i\bar{\sigma})^2 \ge 0,$$

due to (2.13). It then follows that $p_{i,b} \ge 0$.

**Case 3.** $x > x^{i,2}$. This can be done by using the same arguments as in Case 2. We omit the details. □

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