METHODS OF SOLUTION OF SECOND ORDER LINEAR EQUATIONS ON TIME SCALES

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Abstract. A time scale, $\mathbb{T}$, is a nonempty, closed subset of the real numbers, $\mathbb{R}$. Several methods of solution exist for second order linear equations on a time scale. An advantage of these methods is that we can obtain solutions on a system comprising of continuous and/or discrete elements. After restricting the time scale to be $\mathbb{R}$, these solutions are equivalent to those obtained using differential equations methods.

1. Introduction

A time scale, denoted by $\mathbb{T}$, is an arbitrary nonempty and closed subset of $\mathbb{R}$. Some examples of time scales include $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $[0, 1]$, and the Cantor set. Time scale theory was introduced by Stefan Hilger in his 1988 doctoral dissertation. Instead of having disparate methods for continuous and discrete calculus, time scale theory unifies the two for a single, more efficient method. It is thus possible to use a single formula for studying the rates of change of a system over continuous and/or discrete intervals. Likewise, it is possible to determine solutions for second order linear equations on a generic time scale. [2] Time scale theory is especially useful when studying systems with discrete and continuous elements in their domains, such as an insect population that is continuous for parts of the year and discrete at other parts of the year. [1] The majority of the material discussed in this paper originates from *Dynamic Equations on Time Scales: An Introduction with Applications* by Martin Bohner and Allan Peterson.

2. Fundamentals of Time Scale Theory

The existence of a derivative at a point in a time scale depends on the type of the point itself. Points are classified according to two major operators: the forward jump and backward jump operators. For an arbitrary time scale $\mathbb{T}$, the forward jump operator, $\sigma : \mathbb{T} \to \mathbb{T}$, is defined to be

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

If the set of all points in $\mathbb{T}$ that are larger than t is empty, then $\inf \emptyset = \sup \mathbb{T}$. If $\mathbb{T}$ has a maximum t, then $\sigma(t) = t$. The backward jump operator, $\rho : \mathbb{T} \to \mathbb{T}$, is

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If the set of all points in $\mathbb{T}$ that are less than t is empty, then $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a minimum t, then $\rho(t) = t$. Using these two operators for $t \in \mathbb{T}$, the point can be classified in the following manner:

- t is *right-scattered* if $\sigma(t) > t$;
- t is *right-dense* if $\sigma(t) = t$;
- t is *left-scattered* if $\rho(t) < t$;
- t is *left-dense* if $\rho(t) = t$;
- t is *isolated* if it is left- and right-scattered: $\rho(t) < t < \sigma(t)$; and
- t is *dense* if it is both left- and right-dense: $\rho(t) = t = \sigma(t)$.

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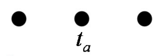
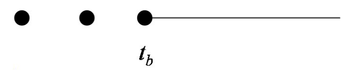
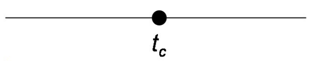


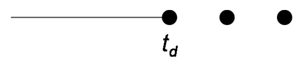
(a) t_a is isolated.



(b) t_b is left-scattered and right-dense.



(c) t_c is dense.



(d) t_d is left-dense and right-scattered.

2.0.1. *Example.* Let $\mathbb{T} = \mathbb{N}_0^3 = \{n^3 : n \in \mathbb{N}_0\}$. Then, for $t \in \mathbb{T}$,

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = (\sqrt[3]{t} + 1)^3.$$

Additionally, for $t \in \mathbb{T}$,

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \begin{cases} 0, & \text{if } t = 0 \\ (\sqrt[3]{t} - 1)^3, & \text{if } t \neq 0 \end{cases}$$

2.1. **The Graininess Function.** For $t \in \mathbb{T}$, the *graininess function*, $\mu : \mathbb{T} \to [0, \infty)$, is defined to be

$$\mu(t) := \sigma(t) - t.$$

The graininess function essentially describes the step size between two consecutive points in $\mathbb{T}$. Oftentimes the differences in results obtained from discrete and continuous calculus stem from the different value of the graininess function evaluated at a given point t. [1]

2.1.1. *Example.* Let $\mathbb{T} = \{\frac{n}{2} : n \in \mathbb{N}_0\}$. Each t in the time scale is right-scattered, and therefore,

$$\sigma(t) = t + \frac{1}{2} > t.$$

Thus, $\mu(t) = \sigma(t) - t = \frac{1}{2}$.

3. The Delta Derivative

The derivative in time scale calculus, called the delta derivative, determines the rates of forward change over a time scale. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative of f at t, $f^\Delta(t)$, is defined to be the number, when it exists, where for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

is true for all $s \in U$. Here, $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$ when $\mathbb{T}$ has a left-scattered maximum m; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Since the delta derivative definition involves the forward jump operator, if the time scale has a left-scattered maximum m, then one cannot jump past this point. Therefore, this point is removed from the set of points used to determine the delta derivative. However, if the time scale does not contain such a left-scattered maximum, then $\mathbb{T}^\kappa$ is equivalent to the time scale. [1]

3.1. **Properties of the Delta Derivative.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$ as defined above. For such a function, the following properties hold:

- If f is delta differentiable at t, then f is continuous at t.
- If t is right-scattered and f is continuous at t, then the delta derivative of f, f^Δ, is defined as follows:

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

- If t is right-dense, then the delta derivative at t is as follows (if and only if the limit exists as a finite number):

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{1}$$

- If f is delta differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

- If $\mathbb{T} = \mathbb{R}$, then the delta derivative is $f'(t)$ from continuous calculus.
- If $\mathbb{T} = \mathbb{Z}$, then the delta derivative is the forward difference, Δf, from discrete calculus. [1]

3.2. **Example.** Suppose $f(t) = t^4 + 3$ where $t \in \mathbb{T}$. If the point t is right-scattered, then using property (ii), the delta derivative is

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{\sigma(t)^4 - t^4}{\sigma(t) - t} = \sigma(t)^3 + t\sigma(t)^2 + t^2\sigma(t) + t^3.$$

If t is right-dense, then after applying equation (1), the result is that $f^{\Delta}(t) = 4t^3$, which is $f'(t)$ from continuous calculus.

4. The Delta Integral

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous if the following conditions are met: f is continuous at right-dense points in $\mathbb{T}$, and the left-sided limits exist and are finite at the left-dense points in $\mathbb{T}$. The set of right-dense continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C_{rd}}$.

4.1. **Theorem.** All right-dense continuous functions have antiderivatives. For a fixed $t_0 \in \mathbb{T}$, the antiderivative F is given by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau$$

for $t \in \mathbb{T}$. [1]

5. The Generalized Exponential Function

As time scale theory generalizes the definition of derivatives, it also includes a generalized definition of the exponential function. The generalized exponential function, denoted $e_p(t, t_0)$, is the solution to the initial value problem

$$\begin{aligned} y^{\Delta}(t) &= p(t)y(t) \qquad t, t_0 \in \mathbb{T} \\ y(t_0) &= 1, \end{aligned} \tag{2}$$

where $1 + \mu(t)p(t) \neq 0$. This condition, the regressivity condition, is required so that the logarithmic expression is not undefined for the function p. The generalized exponential function is thus

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \frac{1}{\mu(\tau)} \mathrm{Log}(1 + \mu(\tau)p(\tau))\Delta\tau\right). \tag{3}$$

For $h > 0$, define $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$, the cylinder transformation, as follows:

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh).$$

If $h = 0$, then the cylinder transformation is defined to be

$$\xi_0(z) = z$$

for all $z \in \mathbb{C}$. Having defined the cylinder transformation, we can now proceed to prove that the generalized exponential function is indeed a solution of the initial value problem (2).

Proof. Let $t_0 \in \mathbb{T}^\kappa$; additionally, assume that

$$y^\Delta(t) = p(t)y(t)$$

is regressive–that is, p is a regressive function. For $t_0 \in \mathbb{T}^\kappa$, the initial condition is met:

$$y(t_0) = e_p(t_0, t_0) = 1.$$

For the event where t is right-scattered (where $\sigma(t) > t$),

$$\begin{aligned}
e_p^\Delta(t, t_0) &= \frac{\exp\left(\int_{t_0}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r\right) - \exp\left(\int_{t_0}^{t} \xi_{\mu(r)}(p(r))\Delta r\right)}{\mu(t)} \\
&= \frac{\exp\left(\int_{t}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r\right) - 1}{\mu(t)} e_p(t, t_0) \\
&= \frac{e^{\xi_{\mu(t)}(p(t))\mu(t)} - 1}{\mu(t)} e_p(t, t_0) \\
&= \xi^{-1}_{\mu(t)}(\xi_{\mu(t)}(p(t)))e_p(t, t_0) \\
&= p(t)e_p(t, t_0).
\end{aligned}$$

On the other hand, let t be a right-dense point, where $\sigma(t) = t$. Then,

$$\begin{aligned}
|y(t) - y(s) - p(t)y(t)(t - s)| &= |e_p(t, t_0) - e_p(s, t_0) - p(t)e_p(t, t_0)(t - s)| \\
&= |e_p(t, t_0)||1 - e_p(s, t) - p(t)(t - s)| \\
&= |e_p(t, t_0)| \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s, t) + \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)(t - s)\right| \\
&\leq |e_p(t, t_0)| \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s, t)\right| \\
&+ |e_p(t, t_0)| \left|\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)(t - s)\right| \\
&\leq |e_p(t, t_0)| \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s, t)\right| \\
&+ |e_p(t, t_0)| \left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right|.
\end{aligned}$$

Then, for an $\epsilon > 0$, the next step is to show the existence of a neighborhood U of t such that the inequality in the delta derivative definition is true. Because $\sigma(t) = t$ and p is a right-dense continuous function,

$$\lim_{r \to t} \xi_{\mu(r)}(p(r)) = \xi_0(p(t)).$$

So, there exists a neighborhood U_α of t such that

$$|\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))| < \frac{\epsilon}{3|e_p(t, t_0)|}$$

for all $\tau \in U_\alpha$. Suppose that $s \in U_\alpha$. It follows that

$$|e_p(t, t_0)| \left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right| < \frac{\epsilon}{3}|t - s|.$$

L'Hospital's rule yields the result

$$\lim_{z \to 0} \frac{1 - z - e^{-z}}{z} = 0,$$

and thus there exists a neighborhood U_β of t such that for $s \in U_\beta$,

$$\left|\frac{1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s, t)}{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau}\right| < \epsilon^*,$$

where the ϵ^* is defined to be

$$\epsilon^* = \min\left\{1, \frac{\epsilon}{1+3|p(t)e_p(t,t_0)|}\right\}.$$

Now, let $s \in U_\alpha \cap U_\beta$. It follows that

$$\begin{aligned}|e_p(t,t_0)|\left|1-\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| &< |e_p(t,t_0)|\epsilon^*\left|\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right| \\ &\leq |e_p(t,t_0)|\epsilon^*\left\{\left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right| + |p(t)||t-s|\right\} \\ &\leq |e_p(t,t_0)|\left|\int_s^t [\xi_{\mu(\tau)}(p(\tau))]\Delta\tau\right| + |e_p(t,t_0)|\epsilon^*|p(t)||t-s| \\ &\leq \frac{\epsilon}{3}|t-s| + |e_p(t,t_0)|\epsilon^*|p(t)||t-s| \\ &\leq \frac{\epsilon}{3}|t-s| + \frac{\epsilon}{3}|t-s| \\ &= \frac{2\epsilon}{3}|t-s|.\end{aligned}$$

□

[1]

5.1. **Properties.** Let p, q be regressive functions and $t, s, t_0 \in \mathbb{T}$. Some important properties of the generalized exponential function in time scale theory include the following:

- $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
- $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$;
- $(p \ominus q)(t) := \frac{p(t)-q(t)}{1+\mu(t)q(t)}$;
- $e^\Delta_{p\ominus q}(t,t_0) = (p-q)\frac{e_p(t,t_0)}{e_q(\sigma(t),t_0)}$; and
- $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$. [1]

6. Sine and Cosine Functions on a Time Scale

Time scale theory also generalizes the definition of the sine and cosine functions. If p is a right-dense continuous function such that $1+\mu p \neq 0$, then the sine and cosine functions are defined as follows:

$$\cos_p(t,t_0) = \frac{e_{ip}(t,t_0) + e_{-ip}(t,t_0)}{2} \qquad \text{and} \qquad \sin_p(t,t_0) = \frac{e_{ip}(t,t_0) - e_{-ip}(t,t_0)}{2i},$$

where $t, t_0 \in \mathbb{T}$.

6.1. **Properties of the Sine and Cosine Functions on a Time Scale.** Let $t, t_0, s \in \mathbb{T}$, μp be a regressive function, and p be right-dense continuous. [1] Then, the following properties hold for the sine and cosine functions on a time scale, $\mathbb{T}$:
1. $\cos^\Delta_p(t,s) = -p\sin_p(t,s)$ and $\sin^\Delta_p(t,s) = p\cos_p(t,s)$.

6.1.1. *Proof of 1.* Let p be right-dense continuous, and suppose $1+\mu p \neq 0$. For $t, s \in \mathbb{T}$, suppose all points are right-scattered. By definition,

$$e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s). \tag{4}$$

Using (4) and the appropriate delta derivative formula, the delta derivative of the cosine function can be calculated in the following manner:

$$\begin{aligned}\cos_p^{\Delta}(t,s) &= \frac{\cos_p(\sigma(t),s)-\cos_p(t,s)}{\mu(t)} = \frac{e_{ip}(\sigma(t),s)+e_{-ip}(\sigma(t),s)}{2\mu(t)} - \frac{e_{ip}(t,s)+e_{-ip}(t,s)}{2\mu(t)} \\ &= \frac{(1+i\mu(t)p(t))e_{ip}(t,s)+(1-i\mu(t)p(t))e_{-ip}(t,s)-e_{ip}(t,s)-e_{-ip}(t,s)}{2\mu(t)} \\ &= \frac{ip(t)(e_{ip}(t,s)-e_{-ip}(t,s))}{2} = -p(t)\sin_p(t,s).\end{aligned}$$

By similar argument, it can be shown that

$$\sin_p^{\Delta}(t,s) = p(t)\cos_p(t,s).$$

2. Euler's Formula in time scale theory is $e_{ip}(t,t_0) = \cos_p(t,t_0) + i\sin_p(t,t_0)$.

6.1.2. *Proof of 2.*

$$\begin{aligned}\cos_p(t,t_0)+i\sin_p(t,t_0) &= \frac{e_{ip}(t,t_0)+e_{-ip}(t,_0)}{2} + i\left(\frac{e_{ip}(t,t_0)-e_{-ip}(t,t_0)}{2i}\right) \\ &= \frac{e_{ip}(t,_0)+e_{-ip}(t,t_0)+e_{ip}(t,t_0)-e_{-ip}(t,t_0)}{2} \\ &= e_{ip}(t,t_0).\end{aligned}$$

7. Solving Second Order Linear Equations

7.1. **Linear Operators and Wronskians.** The second order linear dynamic equation is

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t), \tag{5}$$

where $p, q,$ and f are right-dense continuous functions. If $p, q,$ and f are right-dense continuous functions such that

$$1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0$$

for all $t \in \mathbb{T}^{\kappa}$, then (5) is said to be regressive. [1]

7.1.1. *Linear Operator.* Let the linear operator, $L : \mathrm{C}^2_{\mathrm{rd}} \to \mathrm{C}_{\mathrm{rd}}$, be defined as follows:

$$L(y(t)) = y^{\Delta\Delta}(t) + p(t)y^{\Delta}(t) + q(t)y(t),$$

for all $t \in \mathbb{T}^{\kappa^2}$. Using this definition for the linear operator, (5) can be re-expressed as

$$L(y(t)) = f(t).$$

7.1.2. *Principle of Superposition.* Since L is a linear operator,

$$L(\alpha y_1 + \beta y_2) = \alpha L(y_1) + \beta L(y_2)$$

is true for all $\alpha, \beta \in \mathbb{R}$ and $y_1, y_2 \in \mathrm{C}^2_{\mathrm{rd}}$. Suppose that y_1 and y_2 solve the homogeneous equation $L(y) = 0$. Then, a linear combination of these solutions,

$$y = \alpha y_1 + \beta y_2,$$

is also a solution to the homogeneous equation by the principle of superposition, where $\alpha, \beta \in \mathbb{R}$. [1]

7.1.3. *Existence-Uniqueness Theorem.* Let (5) be regressive, and fix $t_0 \in \mathbb{T}^{\kappa}$. Then, the initial value problem

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t),\ y(t_0) = c_1,\ y^{\Delta}(t_0) = c_2,$$

where c_1 and c_2 are real constants, has a unique solution that is defined for all points in the time scale, $\mathbb{T}$.

7.1.4. *Definition.* Suppose that y_1 and y_2 are solutions to the homogeneous dynamic equation

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = 0,\ y(t_0) = c_1,\ y^{\Delta}(t_0) = c_2.$$

Then, by the principle of superposition,

$$y(t) = \alpha y_1(t) + \beta y_2(t)$$

is also a solution to the homogeneous equation. Then, the next step is to match the new solution to the prescribed initial conditions; therefore, it must be the case that

$$c_1 = y(t_0) = \alpha y_1(t_0) + \beta y_2(t_0)$$

and

$$c_2 = y^{\Delta}(t_0) = \alpha y_1^{\Delta}(t_0) + \beta y_2^{\Delta}(t_0).$$

The resulting system of equations is hence

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1^{\Delta}(t_0) & y_2^{\Delta}(t_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{6}$$

If the 2×2 matrix in the system (6) is invertible, then the system has a unique solution. The Wronskian, $W = W(y_1, y_2)$, where y_1 and y_2 are differentiable functions, is defined as follows:

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1^{\Delta}(t) & y_2^{\Delta}(t) \end{pmatrix}.$$

Suppose that y_1 and y_2 solve the homogeneous equation

$$L(y) = 0.$$

Then, y_1 and y_2 generate a set of solutions if the Wronskian is non-zero for all $t \in \mathbb{T}^{\kappa}$, similar to the Wronskian counterpart in differential equations. [1]

7.2. **Solving Second Order Linear Dynamic Homogeneous Equation with Constant Coefficients.** Consider the initial value problem

$$ay^{\Delta\Delta} + by^{\Delta} + cy = 0, \qquad y(t_0) = \alpha, \qquad y^{\Delta}(t_0) = \beta, \tag{7}$$

where $a, b, c \in \mathbb{R}$. Then, the next step is to find $\lambda \in \mathbb{C}$ such that

$$y(t) = e_{\lambda}(t, t_0)$$

is a solution of (7) and

$$1 + \lambda\mu(t) \neq 0 \text{ for } t \in \mathbb{T}^{\kappa}.$$

Letting $y(t) = e_{\lambda}(t, t_{,0})$, (7) becomes

$$ay^{\Delta\Delta}(t) + by^{\Delta}(t) + cy(t) = (\lambda^2 a + \lambda b + c)e_{\lambda}(t, t_0).$$

Due to the fact that $e_{\lambda}(t, t_0)$ does not vanish, the only way for (7) to be true is if

$$a\lambda^2 + b\lambda + c = 0. \tag{8}$$

Finding the roots of the characteristic equation determines the solution of (7), and the solution depends on whether the roots are distinct, repeated, or complex. [2]

7.2.1. *Case 1: Distinct Roots.* If the characteristic equation produces distinct roots, λ_1 and λ_2, then the general solution is

$$y(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0),$$

where c_1, c_2 are constants to be determined from the initial conditions.
Example. Let $t, t_0 \in \mathbb{T}$, and solve the following dynamic equation:

$$y^{\Delta\Delta} - 6y^{\Delta} + 8y = 0.$$

The characteristic equation is

$$\lambda^2 - 6\lambda + 8 = 0,$$

where the roots are $\lambda_1 = 2$ and $\lambda_2 = 4$. Therefore, the solution is of the form

$$y(t) = c_1 e_2(t, t_0) + c_2 e_4(t, t_0),$$

where the constants are to be determined from initial conditions.

7.2.2. *Case 2: Double Root.* If the characteristic equation has a repeated root, r, then the general solution is

$$y(t) = c_1 e_r(t, t_0) + c_2 e_r(t, t_0) \int_{t_0}^{t} \frac{1}{1 + \mu(\tau) r} \Delta\tau.$$

Example. Let $t, t_0 \in \mathbb{T}$, and solve the following dynamic equation:

$$y^{\Delta\Delta} + 4y + 4 = 0.$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

Therefore, the repeated root is $\lambda = -2$, and the general solution is

$$y(t) = c_1 e_{-2}(t, t_0) + c_2 e_{-2}(t, t_0) \int_{t_0}^{t} \frac{1}{1 - 2\mu(\tau)} \Delta\tau,$$

where $\mu(t)$ depends on the time scale and the constants are to be determined from initial conditions.

7.2.3. *Case 3: Complex Roots.* In the event that the characteristic equation has complex roots $\alpha \pm \beta i$, then the solution is given by

$$y(t) = c_1 e_{\alpha}(t, t_0) \cos_{\frac{\beta}{1+\alpha\mu(t)}}(t, t_0) + c_2 e_{\alpha}(t, t_0) \sin_{\frac{\beta}{1+\alpha\mu(t)}}(t, t_0),$$

where c_1, c_2 are real constants to be determined from the initial conditions.
Example. Let $t, t_0 \in \mathbb{T}$, and determine the general solution to the dynamic equation

$$y^{\Delta\Delta} - 6y^{\Delta} + 25y = 0.$$

The characteristic equation,

$$\lambda^2 - 6\lambda + 25 = 0,$$

has roots $\lambda = 3 \pm 4i$, and so the general solution to the dynamic equation is

$$y(t) = c_1 e_3(t, t_0) \cos_{\frac{4}{1+3\mu(t)}}(t, t_0) + c_2 e_3(t, t_0) \sin_{\frac{4}{1+3\mu(t)}}(t, t_0).$$

[2]

7.3. **Method of Variation of Parameters.** The method of variation of parameters can be used to determine the particular solution for a nonhomogeneous equation of the form

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = g(t). \tag{9}$$

The objective is to find a particular solution of (9), which is

$$y(t) = \alpha(t)y_1(t) + \beta(t)y_2(t),$$

where the functions $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = 0.$$

The functions $\alpha(t)$ and $\beta(t)$ are found by solving the following system of equations:

$$\alpha^{\Delta}(t)y_1(\sigma(t)) + \beta^{\Delta}(t)y_2(\sigma(t)) = 0$$
$$\alpha^{\Delta}(t)y_1^{\Delta}(\sigma(t)) + \beta^{\Delta}(t)y_2^{\Delta}(\sigma(t)) = g(t).$$

Then, solving for $\alpha^{\Delta}(t)$ and $\beta^{\Delta}(t)$, the system becomes

$$\begin{pmatrix} \alpha^{\Delta}(t) \\ \beta^{\Delta}(t) \end{pmatrix} = \frac{1}{W(y_1,y_2)(\sigma(t))} \begin{pmatrix} y_2^{\Delta}(\sigma(t)) & -y_2(\sigma(t)) \\ -y_1^{\Delta}(\sigma(t)) & y_1(\sigma(t)) \end{pmatrix} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

It follows from the system that

$$\alpha^{\Delta}(t) = -\frac{y_2(\sigma(t))g(t)}{W(y_1,y_2)(\sigma(t))} \text{ and } \beta^{\Delta}(t) = \frac{y_1(\sigma(t))g(t)}{W(y_1,y_2)(\sigma(t))}.$$

The next step is to solve for α and β by integrating. Then, for a fixed $t_0 \in \mathbb{T}^{\kappa}$, the solution of the initial value problem

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = g(t), \qquad y(t_0) = c_1, \qquad y^{\Delta}(t_0) = c_2$$

is thus

$$y(t) = \alpha_0 y_1(t) + \beta_0 y_2(t) + \int_{t_0}^{t} \frac{y_1(\sigma(\tau))y_2(t) - y_2(\sigma(\tau))y_1(t)}{W(y_1,y_2)(\sigma(\tau))} g(\tau)\Delta\tau,$$

and the constants α_0 and β_0 are calculated as follows:

$$\alpha_0 = \frac{y_2^{\Delta}(t_0)c_1 - y_2(t_0)c_2}{W(y_1,y_2)(t_0)} \text{ and } \beta_0 = \frac{y_1(t_0)c_2 - y_1^{\Delta}(t_0)c_1}{W(y_1,y_2)(t_0)}.$$

[2]

7.3.1. *Example.* Use the variation of parameters method to solve the dynamic equation

$$y^{\Delta\Delta} - 6y^{\Delta} + 8y = e_3(t,t_0),$$

where $t, t_0 \in \mathbb{T}$. For the homogeneous problem, the characteristic equation is

$$\lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0,$$

and the roots are $\lambda_1 = 2$ and $\lambda_2 = 4$. Therefore, the solution to the homogeneous problem can be expressed as

$$y_h(t) = c_1 e_2(t,t_0) + c_2 e_4(t,t_0).$$

Now, to find the particular solution, the $\alpha(t)$ and $\beta(t)$ are determined such that the following system is true:

$$\begin{aligned} \alpha^{\Delta}(t)e_2(\sigma(t),t_0) + \beta^{\Delta}(t)e_4(\sigma(t),t_0) \qquad &= 0 \\ 2\alpha^{\Delta}(t)e_2(\sigma(t),t_0) + 4\beta^{\Delta}(t)e_4(\sigma(t),t_0) \qquad &= e_3(t,t_0). \end{aligned}$$

Using elimination to solve $\beta^{\Delta}(t)$, the result is that

$$\beta^{\Delta}(t) = \frac{e_3(t,t_0)}{2e_4(t,t_0)}.$$

After integrating for β, it follows that

$$\beta(t) = -\frac{1}{2} e_{\frac{-1}{1+4\mu(t)}}(t,t_0).$$

Then, the same method is applied to determine α. After solving for α^{Δ}, the result is

$$\alpha^{\Delta}(t) = \frac{e_3(t,t_0)}{-2e_2(\sigma(t),t_0)}.$$

After integrating, $\alpha(t)$ is as follows:

$$\alpha(t) = \frac{-1}{2} e_{\frac{1}{1+2\mu(t)}}(t,t_0).$$

Therefore, the particular solution is

$$\begin{aligned} y_p(t) \quad &= \alpha(t)e_2(t,t_0) + \beta(t)e_4(t,t_0) \\ &= \frac{-1}{2} e_{\frac{1}{1+2\mu(t)}}(t,t_0)e_2(t,t_0) + \frac{-1}{2} e_{\frac{-1}{1+4\mu(t)}}(t,t_0) \\ &= \frac{-1}{2} e_3(t,t_0) + \frac{-1}{2} e_3(t,t_0) \\ &= -e_3(t,t_0). \end{aligned}$$

The general solution is therefore

$$y(t) = c_1 e_2(t,t_0) + c_2 e_4(t,t_0) - e_3(t,t_0).$$

7.4. **Method of Solving Euler-Cauchy Equations.** We now consider the Euler-Cauchy dynamic equation

$$t\sigma(t)y^{\Delta\Delta} + aty^{\Delta} + by = 0,$$

where $a, b \in \mathbb{R}$ on a time scale $\mathbb{T}$. Assume that the time scale is such that

$$\mathbb{T} \subset (0, \infty)$$

and that

$$t\sigma(t) - at\mu(t) + b\mu^2(t) \neq 0.$$

The characteristic equation of the Euler-Cauchy equation is

$$\lambda^2 + (a-1)\lambda + b = 0.$$

The general solution depends on whether the roots of the characteristic equation are distinct, repeated, or complex.

7.4.1. *Case 1: Distinct Roots.* If the characteristic equation has distinct roots λ_1 and λ_2, then the general solution is defined by

$$y(t) = \alpha e_{\frac{\lambda_1}{t}}(t,t_0) + \beta e_{\frac{\lambda_2}{t}}(t,t_0).$$

7.4.2. *Case 2: Double Root.* If a double root, α, satisfies the characteristic equation, then the general equation is given by

$$y(t) = c_1 e_{\frac{\alpha}{t}}(t,t_0) + c_2 e_{\frac{\alpha}{t}}(t,t_0) \int_{t_0}^{t} \frac{1}{\tau + \alpha\mu(\tau)} \Delta\tau$$

for $t \in \mathbb{T}$. [1]

7.4.3. *Case 3: Complex Roots.* In the event that the characteristic equation has complex roots $\lambda_{1,2} = \alpha \pm \beta i$, then the general solution is

$$y(t) = c_1 e_{\frac{\alpha}{t}}(t,t_0) \cos_{\frac{\beta}{t+\alpha\mu(t)}}(t,t_0) + c_2 e_{\frac{\alpha}{t}}(t,t_0) \sin_{\frac{\beta}{t+\alpha\mu(t)}}(t,t_0).$$

[3]

Example. Solve the following second order Euler-Cauchy dynamic equation for a general time scale, $\mathbb{T}$:

$$t\sigma(t)y^{\Delta\Delta} - 5ty^{\Delta} + 8y = 0.$$

The characteristic equation is therefore

$$\lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0,$$

and the roots are $\lambda_1 = 2, \lambda_2 = 4$. Since the roots are distinct, the general solution is

$$y(t) = c_1 e_{\frac{2}{t}}(t, t_0) + c_2 e_{\frac{4}{t}}(t, t_0),$$

where c_1, c_2 are real constants to be determined from initial conditions.

7.5. **Method of Reduction of Order.** Consider the equation of the form

$$y^{\Delta\Delta} - 2py^{\Delta} + p^2 y = 0, \qquad y(t_0) = c_1, \qquad y^{\Delta}(t_0) = c_2$$

where p is a constant. One solution of the equation is

$$y_1(t) = e_p(t, t_0),$$

since

$$y_1^{\Delta\Delta} - 2py_1^{\Delta} + p^2 y_1 = p^2 e_p(t, t_0) - 2p^2 e_p(t, t_0) + p^2 e_p(t, t_0) = 0.$$

The next step is to determine a second linearly independent solution of the equation of the form

$$y_2(t) = v(t) e_p(t, t_0).$$

The second linearly independent solution is given by

$$y_2(t) = e_p(t, t_0) \int_{t_0}^{t} \frac{1}{1 + \mu(\tau)p} \Delta\tau.$$

Therefore, the general solution is

$$y(t) = e_p(t, t_0) \left[c_1 + (c_2 - pc_1) \int_{t_0}^{t} \frac{1}{1 + \mu(\tau)p} \Delta\tau \right].$$

Now, suppose that $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$; therefore,

$$y(t) = e^{pt} \left[c_1 + (c_2 - pc_1) \int_0^t 1 d\tau \right] = c_1 e^{pt} + (c_2 - pc_1) t e^{pt},$$

which is the solution determined using differential equation methods.

7.6. **Method of Factoring.** The method of factoring determines the solution for equations that can be factored as follows:

$$(y^{\Delta} - py)^{\Delta}(t) - q(t)[y^{\Delta}(t) - p(t)y(t)] = 0, \tag{10}$$

where p, q are regressive. Let y be a solution of (10) and define $v(t)$ as

$$v(t) := y^{\Delta}(t) - p(t)y(t). \tag{11}$$

Therefore, the equation (10) can be expressed as

$$v^{\Delta}(t) - q(t)v(t) = 0.$$

The result of solving this equation is that

$$v(t) = c_2 e_q(t, t_0), \tag{12}$$

and inserting (12) into the equation (11), the result is

$$y^{\Delta} - p(t)y = c_2 e_q(t, t_0).$$

The general solution of the dynamic equation is then

$$y(t) = c_1 e_p(t, t_0) + c_2 e_p(t, t_0) \int_{t_0}^{t} e_p(t, \sigma(\tau)) e_q(\tau, t_0) \Delta\tau.$$

[1]

Example. Using the method of factoring, determine the solution to the dynamic equation

$$y^{\Delta\Delta} - (5 + t)y^{\Delta} + 5ty = 0$$

The equation can be factored as

$$(y^{\Delta} - 5y)^{\Delta} - t(y^{\Delta} - 5y) = 0,$$

and so the general solution is

$$y(t) = c_1 e_5(t, t_0) + c_2 e_5(t, t_0) \int_{t_0}^{t} e_5(t, \sigma(\tau)) e_t(\tau, t_0) \Delta\tau.$$

7.7. **Annihilator Method.** The annihilator method is for solving equations of the form

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = f(t),$$

where α and β are constants and $f(t)$ is a function that can be annihilated. The identity operator is defined to be

$$D^0 = I,$$

and for $n \in \mathbb{N}$,

$$D^n f(t) = f^{\Delta^n}(t).$$

A function $f : \mathbb{T} \to \mathbb{R}$ can be annihilated if there exists an operator of the form

$$\alpha_n D^n + \alpha_{n-1} D^{n-1} + ... + \alpha_0 I,$$

where

$$(\alpha_n D^n + \alpha_{n-1} D^{n-1} + ... + \alpha_0 I) f(t) = 0.$$

Some common functions and their annihilators are listed in the table below:

Function	Annihilator
1	D
t	D^2
$e_\alpha(t, t_0)$	$D - \alpha I$
$\sin_\alpha(t, t_0), \alpha > 0$	$D^2 + \alpha^2 I$
$\cos_\alpha(t, t_0), \alpha > 0$	$D^2 + \alpha^2 I$

[1]

7.7.1. *Example of Solving with Annihilator Method.* Solve the following equation:

$$y^{\Delta\Delta}(t) - 3y^{\Delta}(t) + 2y(t) = e_2(t, t_0), \tag{13}$$

where $t, t_0 \in \mathbb{T}$. Expressing the derivatives in (13) in operator form results in

$$(D^2 - 3D + 2I) y(t) = e_2(t, t_0). \tag{14}$$

Then, (14) can be factored to determine the roots:

$$(D - I)(D - 2I)(D - 2I) y(t) = 0.$$

The general solution is thus

$$y(t) = \alpha e_1(t, t_0) + \beta e_2(t, t_0) + \lambda e_2(t, t_0) \int_{t_0}^{t} \frac{1}{1 + 2\mu(\tau)} \Delta\tau.$$

As seen in the general equation, the particular solution is

$$y_p(t) = \lambda e_2(t, t_0) \int_{t_0}^{t} \frac{1}{1 + 2\mu(\tau)} \Delta\tau,$$

and substituting in (13) produces

$$y_p^{\Delta\Delta}(t) - 3y_p^{\Delta}(t) + 2y_p(t) = \lambda e_2(t, t_0) = e_2(t, t_0).$$

Therefore, $\lambda = 1$. The general solution is thus

$$y(t) = \alpha e_1(t, t_0) + \beta e_2(t, t_0) + e_2(t, t_0) \int_{t_0}^{t} \frac{1}{1 + 2\mu(\tau)} \Delta\tau.$$

7.7.2. *Comparison of Solution in* $\mathbb{T}$ *and in* $\mathbb{R}$. Considering the equation

$$y^{\Delta\Delta}(t) - 3y^{\Delta}(t) + 2y(t) = e_2(t, t_0),$$

the solution in $\mathbb{T}$ is

$$y(t) = \alpha e_1(t, t_0) + \beta e_2(t, t_0) + e_2(t, t_0) \int_{t_0}^{t} \frac{1}{1 + 2\mu(\tau)} \Delta\tau.$$

However, when $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$. Assuming $t_0 = 0$, the result is

$$y(t) = \alpha e^t + \beta e^{2t} + e^{2t} \int_0^t 1 d\tau = \alpha e^t + \beta e^{2t} + te^{2t},$$

which is the result from applying differential equation methods.

8. Conclusion

The advantage of time scale theory is the generalized approach to problems. Methods of solution for second order linear equations on a time scale yield the same result as methods of solution from differential equations if the time scale is restricted to be $\mathbb{R}$. It is also possible to determine solutions of an equation on a domain comprising of a combination of discrete and continuous elements.

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