Bisections and Reflections: A Geometric Investigation

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Abstract

In this paper we explore a geometric problem involving reflections over the angle bisectors of a triangle. Starting with a point on a side of an arbitrary triangle bisect one of the angles on either side of the point and reflect the point across this line. Continuing in this manner, being sure to travel the same direction that we started around the triangle, leads to some interesting conclusions. In particular we will show that any point will return to its original position in at most six reflections. Furthermore, there is a special point on each side of every triangle which has a return time of exactly three reflections. This point has a special geometric significance. In addition to these conjectures, we present some other interesting geometric observations.

1 Introduction

This problem was first introduced as a challenge problem to high school geometry students. After receiving no responses from students, it was then taken to the collegial level. We were given this problem to investigate with the intent of generalizing the results to a broader question. This question being, what happens when we have a polygon with n sides? Before introducing the problem itself, we will recall some basic geometry definitions which will be used throughout this paper.

Definition. *The* **angle bisector** *is the line or line segment that divides an angle into two congruent parts.*

Definition. *The* **reflection** *through a line in the plane is the transformation that takes each point in the plane to its mirror image with respect to the line. A corresponding point must be the same distance from the line of reflection as its original point.*

Definition. *The* **incenter** *of a triangle is the point of intersection of the angle bisectors and is the center of the incircle of the triangle.*

Definition. *An* **apothem** *of a triangle is the line segment connecting the incenter and the side such that the segment and side are perpendicular.*

*Directed by Dr. Ron Taylor.

Now that we are familiar with the basic definitions let's take a look at the problem.

Take any triangle $\triangle ABC$ and a point, P_1, on any one side of that triangle as shown in Figure 1 below. Now moving in a counterclockwise or clockwise direction, construct the bisector of the adjacent angle. Next reflect the original point over this angle bisector, forming point P_2. This can be seen in Figure 2 below. Continue with this process of bisecting angles and reflecting points making sure to stay consistent with the direction in which you are traveling around the triangle. Throughout this paper, a reflection across an angle bisector will always be a point on the line containing the side of the triangle. See the figures below.

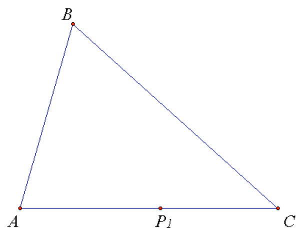

Figure 1

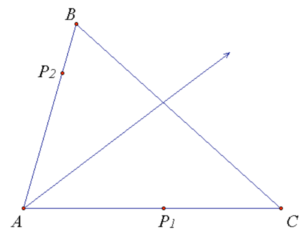

Figure 2

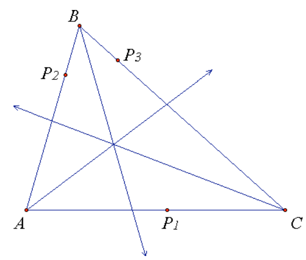

Figure 3

We began approaching this problem systematically, experimenting with different kinds of triangles including acute, obtuse, right, scalene, equilateral, and isosceles triangles. We also adjusted the position of our original point, P_1, on the triangle and noted the different outcomes. In order to facilitate gathering results of this exploration, we utilized Geometer's Sketchpad, which is a dynamic software program that allows a user to construct geometric figures and manipulate them. After conducting several tests we were able to make some conjectures. Three of these conjectures are listed below.

Conjecture (1). *It takes at most six reflections for a point to return to its original position on a triangle.*

Conjecture (2). *There is exactly one point on every side of a triangle that takes exactly three reflections to return to its original position. This point is the point of intersection between the apothem and any side of the triangle.*

Conjecture (3). *The points of reflection lie on a circle with the center being the incenter of the triangle.*

These conjectures are true, and, the proofs are in Section 2.

2 Explorations

The first thing we explored was the positions that a point took when reflected across the angle bisectors of a triangle. After using Geometer's Sketchpad extensively we discovered that a point

returns back to its original position after six reflections.
The proof of this conjecture is relatively straightforward, relying on the definitions from Section 1 and some algebra.

Proposition (1). *It takes at most six reflections for a point to return to its original position on the triangle.*

Proof. Let $\triangle ABC$ be any triangle with sides $\overline{BC}, \overline{AC}$ and $\overline{AB}$ and let P_1 be a point on any side of that triangle. Without loss of generality we will assume that P_1 is on side $\overline{AC}$. Now let the distance from P_1 and the vertex A be x. Then the distance from P_1 and the other vertex on that side, C, is $\overline{AC} - x$. See Figure 4 below. Now bisect $\angle CAB$ and reflect P_1 over this bisector, resulting in P_2 on side $\overline{AB}$. Since reflections preserve distance from points on the line of intersection, we know that P_2 is also x units away from A. So the distance from P_2 to B is $\overline{AB} - x$ as shown in Figure 5. Repeating our process of bisection, bisect $\angle ABC$ and reflect P_2 over this bisector resulting now in P_3. Again since reflections preserve distance, P_3 is $\overline{AB} - x$ units away from B. Continuing with this process, we will get to P_6, which will be $\overline{BC} - [\overline{AB} - [\overline{AC} - [\overline{BC} - (\overline{AB} - x)]]]$ units away from C. Reflecting this sixth time, will give us P_7, which will be the same distance from C as P_6. Furthermore, P_7 will be on side $\overline{AC}$ on the triangle. However, simplifying algebraically gives us

$$\begin{aligned}
\overline{BC} - [\overline{AB} - [\overline{AC} - [\overline{BC} - (\overline{AB} - x)]]] &= \overline{BC} - [\overline{AB} - [\overline{AC} - [\overline{BC} - \overline{AB} + x]]] \\
&= \overline{BC} - [\overline{AB} - [\overline{AC} - \overline{BC} + \overline{AB} - x]] \\
&= \overline{BC} - [\overline{AB} - \overline{AC} + \overline{BC} - \overline{AB} + x] \\
&= \overline{BC} - \overline{AB} + \overline{AC} - \overline{BC} + \overline{AB} - x \\
&= \overline{AC} - x.
\end{aligned}$$

Thus we have returned to our original point and P_7 is actually P_1, as shown in Figure 6. Therefore it takes six reflections to return to the original point on the triangle.

□

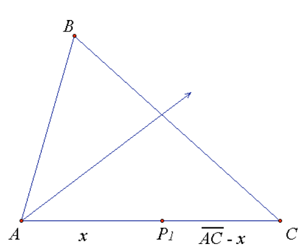

Figure 4

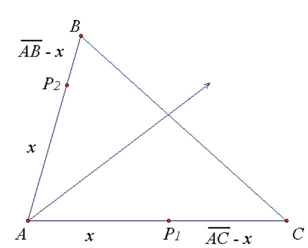

Figure 5

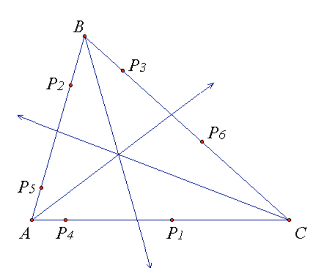

Figure 6

After discovering that in any triangle at most six reflections returns to the original point, we wanted to find out if there were specific points on the triangle which gave a return time of less than six. Since we looked at all different types of triangles, we discovered that in an equilateral triangle if the original point is the midpoint of any side, it has a return time of exactly three. The proof of this particular observation also relies on the definition of reflection and the characteristics of an equilateral triangle. Since reflections preserve distance and the distance from any midpoint to a vertex is the same, the proof of this observation is fairly simple and it follows directly from the previous proof.

This discovery led us to inquire if on any given triangle, non-equilateral as well, there is a specific point that has a return time of exactly three. After exploring with many triangles on Geometer's Sketchpad, we discovered that in fact, this point does exist.

The following conjecture is a special case of the previous one and the proof is similar. This proof uses both algebra and geometry. The algebraic portion of the proof simply shows that there is a point on every triangle that has a return time of exactly three. The geometric portion, however, shows that this point is precisely the point of intersection between the apothem and any side of the triangle. In order to prove this conjecture we will need to use the following lemma.

Lemma. *[Triangle Inequality] The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.*

Proposition (2). *There is exactly one point on every side of a triangle that takes exactly three reflections to return to its original position. This point is the point of intersection between the apothem and any side of the triangle.*

Proof. Assume $\triangle ABC$ is any triangle with sides $\overline{BC}, \overline{AC}$, and $\overline{AB}$ and let P_1 be a point on any side. Without loss of generality suppose P_1 is on $\overline{AC}$. Algebraically, if we want P_1 to return to its original position after only three reflections, the original point, P_1, must be the same distance from vertex C as our third point, P_3 as shown in Figure 7.

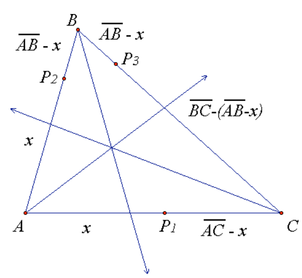

Figure 7

We know that no matter where point P_3 is on side $\overline{BC}$ it will be $\overline{BC} - (\overline{AB} - x)$ units away from our vertex C. We want $\overline{BC} - (\overline{AB} - x)$ to equal $\overline{AC} - x$. Setting up an equation, we can solve to find out what x must be in order to satisfy this condition. So we must have

$$\begin{aligned} \overline{BC} - (\overline{AB} - x) &= \overline{AC} - x \\ \overline{BC} - \overline{AB} + x &= \overline{AC} - x \\ 2x &= \overline{AC} + \overline{AB} - \overline{BC} \\ x &= \frac{\overline{AC} + \overline{AB} - \overline{BC}}{2}. \end{aligned}$$

Using the Triangle Inequality, we know that $\overline{AC} + \overline{AB} > \overline{BC}$ so $\frac{\overline{AC}+\overline{AB}-\overline{BC}}{2} > 0$ no matter what the lengths of the triangle may be. Thus if $x = \frac{\overline{AC}+\overline{AB}-\overline{BC}}{2}$, we know it will take exactly three

reflections to return to our original point, P_1. Furthermore, this point can be found on any side of the triangle.
We will now show that P_1 must be the point of intersection between the apothem of $\overline{AC}$ and $\overline{AC}$. Let P_1 be the point of intersection between the apothem of $\overline{AC}$ and $\overline{AC}$ and let I be the incenter of the triangle. Then $\angle IP_1A = 90°$, by the definition of an apothem. See Figure 8 below.
After bisecting $\angle CAB$ and reflecting P_1, we know that $\overline{AP_1} \cong \overline{AP_2}$, because reflections preserve distance from points on the line of intersection. In addition $\angle P_1AI \cong \angle P_2AI$, by the definition of angle bisector. Using the reflexive property, $\overline{AI} \cong \overline{AI}$. It follows that $\triangle P_1AI \cong \triangle P_2AI$ by Side-Angle-Side. This can be seen in Figure 9 below.

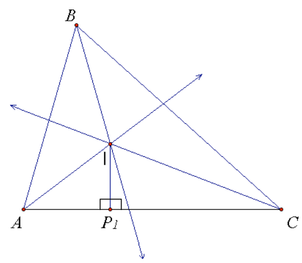

Figure 8

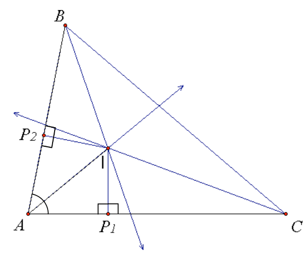

Figure 9

Thus we know that $\angle AP_2I = 90°$ since corresponding parts of congruent triangles are congruent. Furthermore, since $\angle AP_2I$ and $\angle BP_2I$ are supplementary, $\angle BP_2I = 90°$. Now reflect P_2 over the bisector of $\angle ABC$, to obtain P_3 at $\overline{BC}$. Using the same argument we know that $\triangle P_2BI \cong \triangle P_3BI$ by Side-Angle-Side. So $\angle BP_3I = 90°$ and $\angle CP_3I = 90°$. Repeating this process a final time we see that the reflection of P_3 across the bisector of $\angle BAC$ is P_1, which is the point of intersection with the apothem.
Hence points on a triangle that have a return time of exactly three reflections are the points of intersection between the apothems and the original sides. □

Investigating the geometric properties of this problem led us to the following proposition. The proof relies heavily on the properties of reflections and the definition given below.

Definition. *A* **circle** *is the locus of all points equidistant from a central point.*

Proposition (3). *The points of reflection lie on a circle with the center being the incenter of the triangle.*

Proof. Let $\triangle ABC$ be any triangle with sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ and let $P_1, \ldots, P_6$ be the points of reflection over the angle bisectors described in the proof of Proposition 1. Let I be the incenter of the triangle as shown in Figure 10 below. Since each angle bisector passes through the incenter of the triangle, we know that I is always on the line of reflection for each point. Since reflections preserve distance from any point on the line of reflection, we know that

$$\overline{P_1I} \cong \overline{P_2I} \cong \overline{P_3I} \cong \overline{P_4I} \cong \overline{P_5I} \cong \overline{P_6I}.$$

Thus each point of reflection on the triangle is the same distance from the incenter, I. Therefore, by definition of a circle, the points $P_1, \ldots, P_6$ lie on a circle with the center being the incenter, I, of the triangle. This is shown in Figure 11 below.

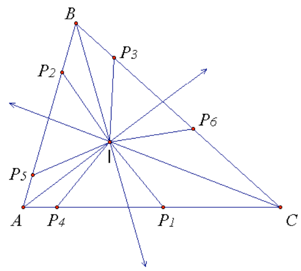

Figure 10

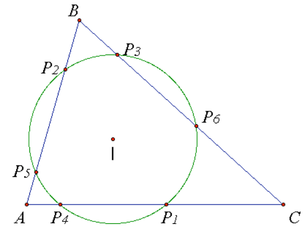

Figure 11

□

3 Conclusion

A natural question that arises from our investigation is "Can we extend this problem to a more general case?" A more general case being a polygon with more than three sides. Using a similar process described in this problem, we will explore this larger problem in hopes of discovering many interesting geometrical properties.

References

[1] J. Berglund & R. Taylor, *A geometric function with a periodic orbit*, preprint, 2005.

[2] Kay, David C., *College Geometry: A Discovery Approach*, Addison Wesley, Boston, MA, 2001.

[3] KCP Technologies, *The Geometer's Sketchpad*, Key College Press, Emeryville, CA.