An Investigation of Continued Fractions

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Abstract: The study of continued fractions has produced many interesting and exciting results in number theory and other related fields of mathematics. Continued fractions have been studied for centuries by many famous mathematicians such as Wallis, Euler, Gauss, Lagrange, Ramanujan, Cauchy, and Khinchin. A connection between continued fractions and the Fibonacci sequence can be revealed by examining functional parameters of various rational functions. This work makes use of existing results concerning continued fractions and Mathematica to explore the relationship between continued fractions and rational functions.

Continued fractions are one of many ways of representing real numbers. The actual term "continued fraction" has been around for 350 years since it was first seen in John Wallis' *Arithmetica Infinitorum* in 1653. A continued fraction is a sequence of integers that represent a real number. These sequences have a strong impact in number theory and have been studied by many mathematicians such as Euler, Gauss, Lagrange, Ramanujan, Cauchy, and Khinchin. Not difficult to understand and innumerable in their uses, continued fractions provide mathematicians with a different way to express the numbers they work with day in and day out.

The finite simple continued fraction representation of a real number x has the form:

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\text{O} + \cfrac{1}{a_n}}}}$$

where $a_1, a_2, a_3 ..., a_n$ are non - negative integers and $a_2, a_3, ..., a_n > 0$.

An abbreviated way of writing this continued fraction is:

$$x = [a_1, a_2, a_3, \ldots, a_n].$$

It is not difficult to obtain the continued fraction form of a rational number; the process very closely resembles that of Euclid's Algorithm for finding the greatest common divisor. The process begins by finding the greatest integer part of the rational number and then determining the fraction remainder, $\frac{p_1}{q_1}$. To continue the fraction, the remaining fraction is replaced by $\frac{1}{\frac{q_1}{p_1}}$. The process is repeated with this new rational number, $\frac{q_1}{p_1}$ and continues until the remaining fraction itself is of the form $\frac{1}{a_n}, a_n \in Z$. An example of this process is shown below:

$$\frac{107}{59} = \mathbf{1} + \frac{48}{59} = \mathbf{1} + \frac{1}{\frac{59}{48}}$$

$$\frac{59}{48} = \mathbf{1} + \frac{11}{48} = \mathbf{1} + \frac{1}{\frac{48}{11}}$$

$$\frac{48}{11} = \mathbf{4} + \frac{4}{11} = \mathbf{4} + \frac{1}{\frac{11}{4}}$$

$$\frac{11}{4} = \mathbf{2} + \frac{3}{4} = \mathbf{2} + \frac{1}{\frac{4}{3}}$$

$$\frac{4}{3} = \mathbf{1} + \frac{1}{\mathbf{3}}$$

So, the continued fraction representation of $\frac{107}{59}$ is:

$$\frac{107}{59} = 1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}}}$$

$$\text{or } \frac{107}{59} = [1,1,4,2,1,3].$$

Using Mathematica, I took a backwards approach to looking at continued fractions of rational functions. Rather than starting with the rational functions themselves, I input continued fraction expansions of the forms [x], [a,x], [a,a,x], …, [a,a,a,…,x] to see the type of generating fractions that were produced. What I found was quite interesting:

- Each reduced form of the continued fraction $[a, a, a, ..., x]$ is of the form:

  $$\frac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)} \text{ for all } n \geq 2, \text{ where } S_0 = 0, S_1 = 1, S_2 = a, S_n = aS_{n-1} + S_{n-2}.$$

- All of the graphs of the continued fractions have a point of concurrency on the line $y = x$ that is of the form:

  $$x = \frac{a + \sqrt{a^2 + 4}}{2}.$$

- Using a power series expansion whose Taylor coefficients are the point of concurrency and its conjugate, an explicit formula for the sequence can be generated.

As mentioned, the initial investigation began by looking at the continued fractions from a backward perspective. By inputting the continued fraction expansion, I was able to analyze the generating fractions that were produced from these expansions. When *a* = 1, the Fibonacci sequence arises in the coefficients of x and the constants in both the numerators and denominators of these generating fractions.

Close inspection shows the Fibonacci sequence appearing in the following pattern:

$\frac{F_{n-1} + F_n(x)}{F_{n-2} + F_{n-1}(x)}$, where $F_0 = 0, F_1 = 1, F_2 = 1, F_n = 1F_{n-1} + F_{n-2}$ and n is the number of elements in the expansion.

There appears to be a similar pattern occurring in all of the successive fractions as well. Investigation concludes that these generating fraction are of the same form as those with the Fibonacci sequence only with a different, yet dramatically similar generating sequence. The fractions all contain a sequence, S_n, such that

$\frac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)}$ where $S_0 = 0, S_1 = 1, S_2 = a$ and $S_n = aS_{n-1} + S_{n-2}$ where *a* is the constant used in the continued fraction expansion and *n* is the number of terms in the expansion.

In the *a* = 1 (Fibonacci) case, the generating fractions were of the form $\frac{F_{n-1} + F_n(x)}{F_{n-2} + F_{n-1}(x)}$. A proof by mathematical induction shows that this is indeed the case.

Theorem: For $n \geq 2$ (n = number of elements in the continued fraction sequence),

$$[1,1,1,\ldots,x] = \frac{F_{n-1} + F_n(x)}{F_{n-2} + F_{n-1}(x)}$$

where $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots, F_n = F_{n-1} + F_{n-2}$.

Proof: (By Induction)

1. Let $n = 2, \dfrac{F_1 + F_2(x)}{F_0 + F_1(x)} = \dfrac{1+x}{x}$

$$[1,x] = 1 + \frac{1}{x} = \frac{1+x}{x}$$

2. Assume $f_n(x) = \dfrac{F_{n-1} + F_n(x)}{F_{n-2} + F_{n-1}(x)}$

Want $f_{n+1}(x) = \dfrac{F_n + F_{n+1}(x)}{F_{n-1} + F_n(x)}$

$$f_{n+1}(x) = f(f_n(x)) = 1 + \frac{1}{f_n(x)}$$

$$= 1 + \frac{1}{\dfrac{F_{n-1} + F_n(x)}{F_{n-2} + F_{n-1}(x)}} = 1 + \frac{F_{n-2} + F_{n-1}(x)}{F_{n-1} + F_n(x)}$$

$$= \frac{F_{n-1} + F_{n-2} + (F_{n-1} + F_n)x}{F_{n-1} + F_n(x)} = \frac{F_n + F_{n+1}(x)}{F_{n-1} + F_n(x)}$$

Therefore, by the principle of mathematical induction,

$$[1,1,1,\ldots,x] = \frac{F_{n-1} + F_n(x)}{F_{n-2} + F_{n-1}(x)} \text{ for all } n \geq 2$$

where $F_0 = 0, F_1 = 1, F_2 = 1, \ldots, F_n = F_{n-1} + F_{n-2}$ and n is the number of terms in the continued fraction expansion. QED

The general case is dealt with in the same manner as the case where $a = 1$. With a proof by mathematical induction, we can see that for any value of a, the generating fraction will be of the form $\dfrac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)}$ where the sequence $S_n = aS_{n-1} + S_{n-2}$ with initial values $S_0 = 0, S_1 = 1, S_2 = a$.

Theorem: For $n \geq 2$ (n = number of elements in the continued fraction sequence),

$$[a, a, a, ..., x] = \frac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)}$$

where $S_0 = 0, S_1 = 1, S_2 = a, ..., S_n = aS_{n-1} + S_{n-2}$.

Proof: (By Induction)

1. Let $n = 2, \frac{S_1 + S_2(x)}{S_0 + S_1(x)} = \frac{1 + ax}{x}$

$$[a, x] = a + \frac{1}{x} = \frac{1 + ax}{x}$$

2. Assume $S_n(x) = \frac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)}$

Want $S_{n+1}(x) = \frac{S_n + S_{n+1}(x)}{S_{n-1} + S_n(x)}$

$$S_{n+1}(x) = S(S_n(x)) = a + \frac{1}{S_n(x)}$$

$$= a + \frac{1}{\frac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)}} = a + \frac{S_{n-2} + S_{n-1}(x)}{S_{n-1} + S_n(x)}$$

$$= \frac{a(S_{n-1} + S_n(x)) + S_{n-2} + S_{n-1}(x)}{S_{n-1} + S_n(x)} = \frac{(aS_{n-1} + S_{n-2}) + (aS_n + S_{n-1})x}{S_{n-1} + S_n(x)}$$

$$= \frac{S_n + S_{n+1}(x)}{S_{n-1} + S_n(x)}$$

Therefore, by the principle of mathematical induction,

$$[a, a, a, ..., x] = \frac{S_{n-1} + S_n(x)}{S_{n-2} + S_{n-1}(x)} \text{ for all } n \geq 2$$

where $S_0 = 0, S_1 = 1, S_2 = a, ..., S_n = aS_{n-1} + S_{n-2}$. QED.

Looking at the plotted generating fractions, a point of concurrency for all of the generating fractions on the line y = x for each value of *a* appears. The point of concurrency can not only be determined using the graph, but analytically as well. Again, we will look at the case when *a*=1 (Fibonacci) and then use similar techniques to show that this holds true for all values of *a*.

We begin by letting $\mathrm{f(x)} = 1 + \frac{1}{\mathrm{x}} = [1, x]$.

Then define the proceeding fractions as such :

$$f_1(x) = 1 + \frac{1}{x} = [1, x],\ f_2(x) = 1 + \cfrac{1}{1 + \cfrac{1}{x}} = [1,1,x], \Lambda\ ,\ f_n(x) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\mathrm{O} + \cfrac{1}{x}}}} = [1,1,1,..., x]$$

Then, to find the point of concurrency : $f(x) = 1 + \frac{1}{x} = x$

$$\mathrm{x} + 1 = \mathrm{x}^2$$

$$0 = \mathrm{x}^2 - \mathrm{x} - 1$$

$$\mathrm{x} = \frac{1 \pm \sqrt{5}}{2}$$

and thus, the point of concurrency is $\mathrm{x} = \frac{1 + \sqrt{5}}{2} = \phi_1$ when $\mathrm{a} = 1$.

The negative solution to the equation is not a possibility as it is apparent that there is not a point of concurrency there. We solve for the concurrency point the same way in the general solution.

Let $\mathrm{s(x)} = \mathrm{a} + \frac{1}{\mathrm{x}} = [a, x]$

And,

$$s_1(x) = a + \frac{1}{x} = [a, x], s_2(x) = a + \cfrac{1}{a + \cfrac{1}{x}} = [a, a, x],..., s_n(x) = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\mathrm{O} + \cfrac{1}{x}}}} = [a, a, a,..., x]$$

To find the point of concurrency :

$$s(x) = a + \frac{1}{x} = x$$

$$ax + 1 = x^2$$

$$0 = \mathrm{x}^2 - \mathrm{ax} - 1$$

$$\mathrm{x} = \frac{\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}}{2}$$

and the point of concurrency is $\mathrm{x} = \frac{\mathrm{a} + \sqrt{\mathrm{a}^2 + 4}}{2} = \phi_a$.

From this we conclude, each continued fraction expansion of the form [a,a,a,...,x] has a point of concurrency for all s_i, i=1,2,…,n, which is the point of intersection for the line y=x and all of the graphs of the function and is of the form $x = \frac{a+\sqrt{a^2+4}}{2}$.

So far these continued fractions have been dealt with exclusively with recursive sequences. However, it is possible to define an explicit rule for the sequence that also relates to the point of concurrency found above by using a power series expansion for a function f(x). The following is the recursive sequence we have been discussing so far:

$s_0(a)=c$, $s_1(a)=d$, $s_{n+2}(a)=s_n(a)+as_{n+1}(a)$.

This gives a sequence with the following terms:

$s_0(a)=c$, $s_1(a)=d$, $s_2(a)=c+ad$, $s_3(a)=d+a(c+ad)=d+ac+a^2d$,

$s_4(a)=c+ad+a(d+ac+a^2d)=c+2ad+a^2c+a^3d$. **….**

To find an explicit rule for the sequence, use the series expansion of the function:

$$f(x)=\sum_{n=0}^{\infty} s_n(a)x^n \text{ or } f(x)=c+dx+\sum_{n=0}^{\infty} s_{n+2}(a)x^{n+2}$$

Then, using the second expansion, solve for f(x) to obtain what we will call the generating function.

$$f(x)-c-dx=\sum_{n=0}^{\infty} s_{n+2}(a)x^{n+2}$$

$$f(x)-c-dx=\sum_{n=0}^{\infty}(s_n(a)+as_{n+1}(a))x^{n+2}$$

$$f(x)-c-dx=\sum_{n=0}^{\infty} s_n(a)x^{n+2}+as_{n+1}(a)x^{n+2}$$

$$f(x)-c-dx=\sum_{n=0}^{\infty} s_n(a)x^{n+2}+\sum_{n=0}^{\infty} as_{n+1}(a)x^{n+2}$$

$$f(x)-c-dx=x^2\sum_{n=0}^{\infty} s_n(a)x^{n}+ax\sum_{n=0}^{\infty} s_{n+1}(a)x^{n+1}$$

$$f(x)-c-dx=x^2 f(x)+ax(f(x)-c)$$

$$f(x)-c-dx=x^2 f(x)+axf(x)-acx$$

$$acx-c-dx=x^2 f(x)+axf(x)-f(x)$$

$$(ac-d)x-c=(x^2+ax-1)f(x)$$

$$\frac{(ac-d)x-c}{x^2+ax-1}=f(x)$$

Now we have the generating function for the sequence $S_n(x)$. We will again begin with the Fibonacci case. Remember that the Fibonacci sequence is the sequence of the form:

$S_n=aS_{n-1}+S_{n-2}$ where a=1.

For this sequence we defined c = 0, d = 1, and a = 1.

$$\text{So, } f(x)=\frac{(ac-d)x-c}{x^2+x-1}$$

$$=\frac{-x}{x^2+x-1}$$

$$=\frac{-x}{(x+\frac{1+\sqrt{5}}{2})(x+\frac{1-\sqrt{5}}{2})}$$

$$=\frac{-x}{(x+\phi)(x+\phi^*)}$$

(where ϕ^* is the conjugate of ϕ).

By using partial fraction to decompose the generating function, we will be able to explicitly see the role that the point of concurrency plays in the sequence itself.

$$\frac{-x}{(x+\phi)(x+\phi^*)} = \frac{A}{x+\phi} + \frac{B}{x+\phi^*}$$

$A + B = -1$ and $A\phi^* + B\phi = 0$

$(B = -1 - A)$

So we have $A\phi^* + (-1-A)\phi = 0$

$$A(\phi^* - \phi) = \phi$$

$$A = \frac{\phi}{(\phi^* - \phi)} = \frac{\phi}{-\sqrt{5}}$$

Then $B = -1 + \frac{\phi}{\sqrt{5}} = \frac{-\sqrt{5}+\phi}{\sqrt{5}} = \frac{\phi^*}{\sqrt{5}}$

$$\begin{aligned} f(x) &= \left(\frac{\phi}{-\sqrt{5}}\right)\left(\frac{1}{\phi + x}\right) + \left(\frac{\phi^*}{\sqrt{5}}\right)\left(\frac{1}{\phi^* + x}\right) \\ &= \left(\frac{1}{-\sqrt{5}}\right)\left(\frac{1}{1+\frac{x}{\phi}}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1}{1+\frac{x}{\phi^*}}\right) \\ &= \left(\frac{1}{-\sqrt{5}}\right)\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\phi^n} + \left(\frac{1}{\sqrt{5}}\right)\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\phi^{*n}} \end{aligned}$$

We can now equate the coefficients to get an explicit formula for $s_n(1)$:

$$\begin{aligned} s_n(1) &= \left(\frac{1}{-\sqrt{5}}\right)\left(\frac{(-1)^n}{\phi^n}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{(-1)^n}{\phi^{*n}}\right) \\ &= \left(\frac{1}{-\sqrt{5}}\right)\left(\frac{(-1)^n \phi^{*n}}{\phi^n \phi^{*n}}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{(-1)^n \phi^n}{\phi^{*n}\phi^n}\right) \\ &= \left(\frac{1}{-\sqrt{5}}\right)\left(\frac{(-1)^n \phi^{*n}}{(-1)^n}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{(-1)^n \phi^n}{(-1)^n}\right) \\ &= \frac{\phi^{*n}}{-\sqrt{5}} + \frac{\phi^n}{\sqrt{5}} \end{aligned}$$

This is the explicit formula for the sequence generated with $a = 1$ (or the Fibonacci sequence). We can now see how the point of concurrency (in this case ϕ, or the Golden

Ratio) related to the original recursive sequence. Once again, we will repeat the steps used to generate the explicit form of the Fibonacci sequence to generate the explicit formula for all previously defined sequences $S_n(a)$.

In general, we defined the sequence to be: $S_n=aS_{n-1}+S_{n-2}$ for all a = 1,2,3,... So in general we have, if c = 0, d = 1, and $a = a$, then

$s_0(a)=0$, $s_1(a)=1$, $s_2(a)=a$

$s_3(a)=aa+1=a^2+1$, ….

Using the same series expansions of the function f(x) as before, for all sequences $S_n(a)$, the generating function will be: $f(x)=\frac{-x}{x^2+ax-1}$. By examining the roots of the denominator we once again see that $x=\frac{-a\pm\sqrt{a^2+4}}{2}$ which was the solution discussed previously when finding the points of concurrency. We can then factor the denominator as shown below:

$$\begin{aligned}
x^2+ax-1 &= \left(x-\frac{-a-\sqrt{a^2+4}}{2}\right)\left(x-\frac{-a+\sqrt{a^2+4}}{2}\right)\\
&= \left(x+\frac{a+\sqrt{a^2+4}}{2}\right)\left(x+\frac{a-\sqrt{a^2+4}}{2}\right)\\
&= (x+\phi_a)(x+\phi_a{}^*)
\end{aligned}$$

$$\text{where } \phi_a=\frac{a+\sqrt{a^2+4}}{2}$$

$$\phi_a{}^*=\frac{a-\sqrt{a^2+4}}{2}$$

Again, by decomposing the generating function we can explicitly see the role the point of concurrency plays in the sequence itself:

$$\frac{-x}{(x+\phi_a)(x+\phi_a*)} = \frac{A}{x+\phi_a} + \frac{B}{x+\phi_a*}$$

$A + B = -1$ and $A\phi_a * + B\phi_a = 0$

$(B = -1 - A)$

So we have $A\phi_a * +(-1-A)\phi_a = 0$

$$A(\phi_a * \text{-} \phi_a) = \phi_a$$

$$A = \frac{\phi_a}{(\phi_a * \text{-} \phi_a)} = \frac{\phi_a}{-\sqrt{a^2+4}}$$

Then $B = -1 + \dfrac{\phi_a}{\sqrt{a^2+4}} = \dfrac{-\sqrt{a^2+4}+\phi_a}{\sqrt{a^2+4}} = \dfrac{\phi_a *}{\sqrt{a^2+4}}$

$$f(x) = \left(\frac{\phi_a}{-\sqrt{a^2+4}}\right)\left(\frac{1}{\phi_a + x}\right) + \left(\frac{\phi_a *}{\sqrt{a^2+4}}\right)\left(\frac{1}{\phi_a * + x}\right)$$

$$= \left(\frac{1}{-\sqrt{a^2+4}}\right)\left(\frac{1}{1+\frac{x}{\phi_a}}\right) + \left(\frac{1}{\sqrt{a^2+4}}\right)\left(\frac{1}{1+\frac{x}{\phi_a *}}\right)$$

$$= \left(\frac{1}{-\sqrt{a^2+4}}\right)\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\phi_a{}^n} + \left(\frac{1}{\sqrt{a^2+4}}\right)\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\phi_a *^n}$$

We can now equate the coefficients to give an explicit formula for $S_n(a)$:

$$s_n(a) = \left(\frac{1}{-\sqrt{a^2+4}}\right)\left(\frac{(-1)^n}{\phi_a{}^n}\right) + \left(\frac{1}{\sqrt{a^2+4}}\right)\left(\frac{(-1)^n}{\phi_a *^n}\right)$$

$$= \left(\frac{1}{-\sqrt{a^2+4}}\right)\left(\frac{(-1)^n \phi_a *^n}{\phi_a{}^n \phi_a *^n}\right) + \left(\frac{1}{\sqrt{a^2+4}}\right)\left(\frac{(-1)^n \phi_a{}^n}{\phi_a *^n \phi_a{}^n}\right)$$

$$= \left(\frac{1}{-\sqrt{a^2+4}}\right)\left(\frac{(-1)^n \phi_a *^n}{(-1)^n}\right) + \left(\frac{1}{\sqrt{a^2+4}}\right)\left(\frac{(-1)^n \phi_a{}^n}{(-1)^n}\right)$$

$$= \frac{\phi_a *^n}{-\sqrt{a^2+4}} + \frac{\phi_a{}^n}{\sqrt{a^2+4}}$$

This is the explicit formula for any sequence $S_n(a)=aS_{n-1}+S_{n-2}$ where $s_0 = 0$, $s_1= 1$, $s_2 = a$ and now we can explicitly see how the point of concurrency ϕ_a relates to the original sequence and the generating fractions for all values of a.

Continued fractions are not difficult to understand, but they do produce interesting results when looked at more intensely. Continued fraction expansions of the form [a,a,a,…,x] were investigated. The results found included a pattern in the continued fraction itself using a recursive formula to find the coefficients, the existence of points of concurrency on the graphs of all of the expansions, and the explicit formula for the recursive formulas produced by these continued fraction expansions.

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