An Application of Analytic Geometry to Designing Machine Parts--and Dresses

Karl Hess
Sinclair Community College
Dayton, OH 45402-1460
Email: Karl.Hess@sinclair.edu

Abstract: This paper presents the solution of an engineering problem that the author was asked to solve. The problem involves creating a flat pattern that could be cut from a piece of sheet metal and rolled to form a tube whose top edge would be contained in a plane that is not perpendicular to the central axis of the tube. A piece of this nature needs to be fabricated whenever two sheet metal tubes must be joined at any angle other than a straight angle.

My brother-in-law called me with an intriguing yet surprisingly simple problem one evening about 20 minutes past 11:00. It was not the first time he had called me looking for help with a math problem, nor the first time that he had called me so late. Still, I was tired, and more than a little annoyed with him. Despite that, being asked for help with a problem has always managed to touch a little corner of my mind that is quite proud of being a mathematician. So I listened.

James is a mechanical engineer who works for a firm that designs commercial bakeries. In the course of redesigning a machine that was not working properly, he had found it necessary to have a tubular part which would be fabricated by rolling a piece of sheet metal into a cylinder. What made the situation challenging was that one end of the tube needed to be cut at an angle that was not perpendicular to the central axis of the tube. It's much easier to cut a piece of sheet metal while it is still flat, so James wondered how he would need to cut the top so that it would have the form he desired after the sheet was rolled into a tube.

James is accustomed to relying on AutoCAD to find the necessary geometric properties of a design. In this case, he was at a loss as to how to apply this software. So were the engineers that he works with. Fortunately, years of mathematical training as an engineering student had left him with the ability to recognize this as a problem in analytic geometry. He just wasn't quite sure how to proceed.

After James had explained the problem to me, he asked whether the appropriate curve might be found by describing the tube with an equation and finding its intersection with a plane having the correct slope. I had had the same initial thought almost immediately after he explained the problem. Just as quickly, I had encountered the same problem that was perplexing him. While it is easy enough to describe the intersection of a cylinder and a plane with parametric equations, neither of us could see what would happen to those equations if the intersecting curve were "unrolled" onto a plane.

After realizing that I didn't know how to "unroll" the curve, I decided to give some thought to what I was really trying to find. All I needed was a function describing the curve as it would be when the sheet was flat. It occurred to me that in this state, it might very well be possible to describe the curve with a two-variable formula, where the variables represent rectangular coordinates. For reasons that should seem clearer later on, I will call the variables s and z rather than the more standard x and y. The variable s measures horizontally along the

ISSN: 154-2286

bottom of the sheet from the bottom-left corner. The variable z measures from the bottom of the sheet up to the curving top edge. The trick is to find the connection between s and z.

Once I began thinking about distances along the bottom of the sheet, the problem took on a life of its own. The pieces began to fall into place one after the other, and the closer I came to finishing the problem, the more I doubted myself. I had expected that the function I was seeking would be more complicated than the one taking shape in front of me.

Since our flat sheet is just an unrolled tube, the total horizontal distance along the bottom would simply be the circumference of the tube. (This is why I chose to use s for distances measured along the bottom of the sheet. They are really arc lengths.) I double-checked with my brother-in-law to make sure that no overlap was necessary in the construction of the tube. There wasn't. The seam was to be bonded with a weld. Therefore, an s-value measured along the bottom of the sheet would correspond to an arc length measured along the bottom of the tube starting at the seam. If you construct the tube by rolling the edges up and making the seam on the side closest to you, then the s-values along the bottom of the sheet correspond to an arc length measured clockwise. If you construct the tube by rolling the edges under the sheet and making the seam on the side farthest from you, then the s-values along the bottom of the sheet correspond to an arc length measured counterclockwise. Either way, you get the same tube. We'll assume the tube was rolled so that the arc length can be measured counterclockwise. That way, the corresponding central angle in the middle of the tube is positive. We'll use θ to represent the central angle in the tube corresponding to s.

Now, assuming that θ is measured in radians, we can say that $\theta = \frac{s}{r}$, where r is the radius of the tube. The radius would be a design parameter, so we can treat it as a constant. Therefore, this formula gives us θ as a function of s. The chain of compositions that will take us from s to z is now underway.

Let's place the rolled-up tube into a right-handed three-dimensional coordinate system. (See Figure 1) We'll let the z-axis be the central axis of the tube, and put the bottom of the tube in the x-y plane. We'll put the slanted top of the tube above the x-y plane. Also, I want the shortest portion of the top edge to be directly above the positive x-axis. This last condition wouldn't be absolutely necessary to construct the tube. Here's why I'm including it. The central angle θ is measured counterclockwise from the positive x-axis, so the arc length s is also measured from the positive x-axis. In the unrolled state, s is measured from the side of the sheet. The side of the sheet is where the seam of the tube will be. Therefore, this positioning puts the seam of the tube in the place where it will be the shortest. Since the seam has to be welded, this seems like the logical place for it.

At this point we need to name the other two design parameters of the tube. We'll let a represent the height of the tube at the shortest point, and b represent the height at the tallest point. The plane which contains the top of the tube is perpendicular to the x-z plane, so finding its formula is equivalent to constructing a line in the x-z plane. You can confirm for yourself that the correct formula is $z = \left(\frac{a-b}{2r}\right)x + \left(\frac{a+b}{2}\right)$. Since a, b, and r are all constant for a given tube, what we now have is z given as a function of x.

Now remember, our goal is to get z as a function of s. We have z as a function of x, and θ as a function of s, so all we need to bridge the gap is to express x as a function of θ.

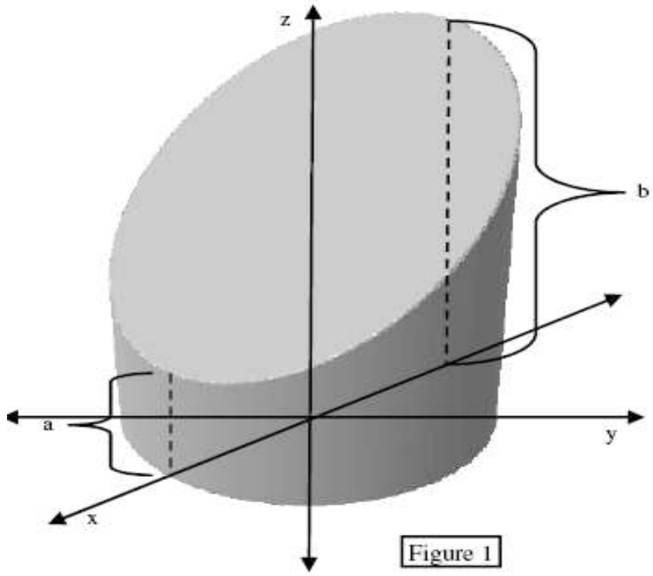


Figure 1

Consider the arbitrary point (x, y, z) in Figure 1 that represents any point on the top edge of the tube. If we project this point into the x-y plane, then we can see that $x = r\cos(\theta)$.

Now we have all the pieces we need to construct the desired function. If we compose the functions $z = \left(\frac{a-b}{2r}\right)x + \left(\frac{a+b}{2}\right)$, $x = r\cos(\theta)$, and $\theta = \frac{s}{r}$, we get:

$$z = \left(\frac{a-b}{2}\right)\cos\left(\frac{s}{r}\right) + \left(\frac{a+b}{2}\right) \qquad \text{(Equation 1)}$$

Now, remembering that s and z are the only variables here, it becomes clear that the curve we were seeking is **just a sine curve!** Could it really be that simple? What's more, the period of this function is $2\pi r$, which is exactly the circumference of the tube. You might also notice that the cosine function has a negative coefficient (b is greater than a, so $a-b$ is negative). Therefore, a period starting at the z-axis would begin and end at the lowest part of the curve.

So, all it takes to create a tube with a slanted top is to cut out a sheet whose top edge is one period of a sine curve? (See Figure 2) Despite the fact that I had derived the result myself, I had to see it to believe it. I drew a similar sine curve using a computer graphing program and then printed it out. I cut along one period of the curve, beginning and ending at low points. I then cut straight down at each end of the period and squared off the bottom. This brought me to the moment of truth. Would the piece of paper really form the type of tube I desired when I rolled it up?

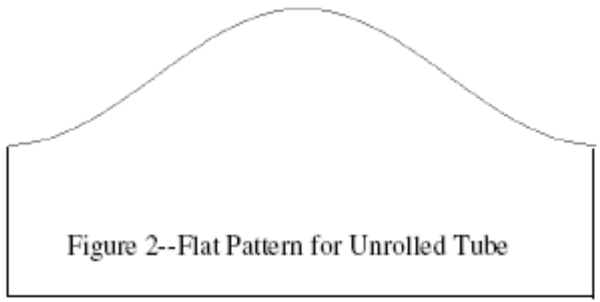


Figure 2--Flat Pattern for Unrolled Tube

I brought the seam together and examined the tube. It did look like the top had been sliced by an imaginary plane. I taped the seam, and then set the tube top-down on a flat surface. It matched up almost perfectly. I could see very little or no light all the way around the top.

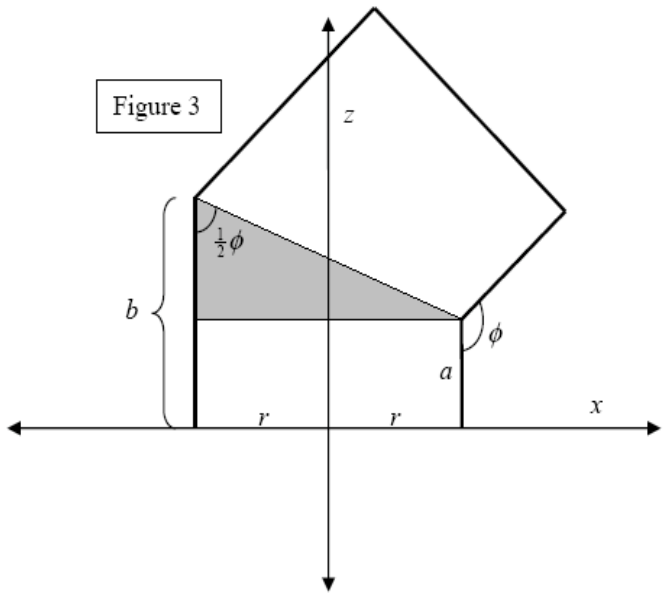
Excited by my discovery, I cut the tape along the seam and hurried off to show it to my wife, who was reading in bed and patiently waiting for me to join her. When I held up the sheet of paper, she put down her book, looked at the paper for about three seconds, and then declared matter-of-factly that I was showing her a sleeve pattern. What?

My wife sews, so I immediately knew what she was referring to, but it took me a minute to see what she meant. A simple dress sleeve is just a tube. At the cuff end, the tube is cut perpendicular to the central axis. However, at the shoulder, the sleeve joins the body of the shirt at an acute angle. The principle is the same as designing a sheet metal tube.

I examined one of the paper templates that my wife uses for sleeves. I'm sure I had seen them lying around before, but hadn't given them much thought. With a new perspective, I immediately recognized the pattern as a sine curve, or at least an approximation to one. In fact, it was one period of a sine curve that began and ended at the lowest point of the curve.

I conducted an internet search, but wasn't able to find any evidence that the connection between sine curves and this type of flat pattern is known, although I assume it must be. Many types of machines and systems of duct work contain circular tubes meeting at various angles. These junctions require tubes whose ends are cut at various angles. These tubes are not always constructed from flat patterns, but they often are. My internet searching indicated that many tailors create these flat patterns by hand on graph paper from detailed measurements and the use of a generic "curve tool". I'm not certain whether engineers also rely on approximation techniques or if they are aware of the sine-curve connection. All I know for certain is that my brother-in-law and his colleagues weren't aware of the connection.

I called James back the same night and explained what I had found. He was able to complete the design and eventually the machine was actually built. I think I got the better end of this deal though. I've been using this problem in an assignment for my trigonometry classes. I give the students specific dimensions for a tube with a slanted top and ask them to find a formula for the necessary flat pattern. I tell them that the curve at the top of the flat pattern is a sine curve, but I leave it to them to determine the exact formula. They don't have too much difficulty in finding the amplitude, period, and shifts or reflections of the curve from the dimensions of the tube. Once they have done so, it simply becomes a problem of creating a formula for a sine curve based on the characteristics of its graph. This is a common "drill" problem in trigonometry courses.



For students to actually derive this result from scratch, they would need to have already covered parametric equations of circles and rectangular equations of planes. Both of these topics are covered in the third quarter of calculus at the college where I teach. I like to incorporate problems such as this into my classes. Of course, not all math students are engineering majors, although in trigonometry and calculus courses usually a plurality are. However, I wish to impress upon all of my students the pervasiveness of mathematics. Complex mathematical ideas can and regularly do emerge from the most unexpected places. It is my admittedly lofty goal for students to leave my courses with the understanding that mathematical skill can conquer much more than degree requirements.

There's one last point I would like to address. One common application of Equation 1 would be to determine how to cut two pipes so that they will join at the angle ϕ. (See Figure 3) In order to get the junction to seal, both pipes must have the same radius, and both pipes will need to be cut in an identical manner. If this is our intention, then it would be useful to rewrite Equation 1 in terms of ϕ. It will still be necessary for either *a* or *b* to be included in the formula to establish the length of each piece of pipe.

Consider the shaded right triangle in Figure 3. It has $\frac{1}{2}\phi$ as an acute angle, and opposite and adjacent sides of lengths *2r* and $b-a$, respectively. This leads to the equation $\tan\frac{1}{2}\phi = \dfrac{2r}{b-a}$. If we solve this equation for *b*, then we can rewrite Equation 1 as

$$\boxed{z = \left(-\frac{r}{\tan\frac{1}{2}\phi}\right)\cos\left(\frac{s}{r}\right) + \left(a + \frac{r}{\tan\frac{1}{2}\phi}\right)}$$

If we solve $\tan\phi = \dfrac{2r}{b-a}$ for *a*, then we can rewrite Equation 1 as

ISSN: 154-2286

$$\boxed{z = \left(-\frac{r}{\tan\frac{1}{2}\phi} \right) \cos\left(\frac{s}{r} \right) + \left(b - \frac{r}{\tan\frac{1}{2}\phi} \right)}$$

These last two boxed formulas allow us to construct the flat pattern formula for the top of each tube using the angle ϕ between the tubes and just one of the length parameters for each tube.