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Rearrangement on Conditionally Convergent Integrals in Analogy to Series

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Abstract

Rearrangements on conditionally convergent series suggests the existence of a similar process for integrals, here also referred to as rearrangement. In this document, a general theorem concerning rearrangement for conditionally convergent integrals is presented, as well as supporting theorems and a corollary to the general theorem. The corollary reads: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with an everywhere negative and monotone increasing derivative. If $\int_1^\infty (-1)^x f(x)dx$ is conditionally convergent, then $\forall z \in \mathbb{C}$, there exists an arrangement on $\int_1^\infty (-1)^x f(x)dx$ such that $z = \int_1^\infty (-1)^x f(x)dx$.

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1 Preliminary Theorems and Lemmas

Note: Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $(-1)^n = e^{in\pi}$.

Definition 1. *For any function $f(x)$ integrable on $[1,\infty)$ such that*

$$\int_1^\infty f(x)dx = \int_0^1 \sum_{j=1}^\infty f(j+u)du$$

a rearrangement on $\int_1^\infty f(x)dx$ is a rearrangement of the terms in the series $\sum_{j=1}^\infty f(j+u)$ for each u on $[0,1]$.

Theorem 1. *Let $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ be positive, real, decreasing sequences which converge to zero, where $a_j \le b_j$ for all j. If*

$$a_j - a_{j+1} \le b_j - b_{j+1} \quad j \in \mathbb{N}$$

then

$$\left|\sum_{j=k+1}^\infty (-1)^j a_j\right| \le \left|\sum_{j=k+1}^\infty (-1)^j b_j\right|.$$

Proof. From the hypothesis, both sequences are decreasing, and therefore $a_j - a_{j+1} > 0$ and $b_j - b_{j+1} > 0$. For $\forall n \in \mathbb{N}$

$$\begin{aligned}\sum_{j=0}^{2n-1} (-1)^j a_{j+1} &= a_l - a_{l+1} + a_{l+2} - a_{l+3} + \ldots + a_{l+2n-2} - a_{l+2n-1}\\ &\le b_l - b_{l+1} + b_{l+2} - b_{l+3} + \ldots + b_{l+2n-2} - b_{l+2n-1}\\ &= \sum_{j=0}^{2n-1} (-1)^j b_{j+l}.\end{aligned}$$

with both sums being positive. Since $a_{2n} \le b_{2n}$, it follows that

$$0 < \sum_{j=0}^{2n} (-1)^j a_{j+l} \le \sum_{j=0}^{2n} (-1)^j b_{j+l}$$

and so $\forall n \in \mathbb{N}$,

$$0 < \sum_{j=0}^{n} (-1)^j a_{j+l} \le \sum_{j=0}^{n} (-1)^j b_{j+l}$$

which implies

$$\left|\sum_{j=l}^{n+l} (-1)^j a_j\right| \le \left|\sum_{j=l}^{n+l} (-1)^j b_j\right|.$$

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Since $a_j, b_j \to 0$ as $j \to \infty$, it follows that both series are convergent. Knowing this, let $l = k+1$ and $n \to \infty$, to yield that

$$\left| \sum_{j=k+1}^{\infty} (-1)^j a_j \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j b_j \right|.$$

□

Theorem 2. *Let f be a positive, decreasing function on $\mathbb{R}^+$, integrable such that $\int_1^\infty (-1)^x f(x)dx$ is conditionally convergent. Then $\sum_{j=1}^\infty (-1)^j f(j+u)$ is conditionally convergent $\forall u \in [0,1]$.*

Proof. Note that, for all $u \in [0,1]$ and for all $x \in \mathbb{R}^+$,

$$\lceil x \rceil + u < x + 2.$$

This implies

$$f(x+2) < f(\lceil x \rceil + u).$$

Let $N \in \mathbb{N}$ such that $N > 2$. Since

$$\int_0^N f(\lceil x \rceil)dx = \sum_{j=1}^N f(j)$$

it follows that

$$\int_2^N f(x)dx < \sum_{j=1}^N f(j+u).$$

Taking the limit that $N \to \infty$ yields that

$$\int_2^\infty f(x)dx < \sum_{j=1}^\infty f(j+u).$$

Since f is integrable on $\mathbb{R}^+$, it follows that $\int_1^2 f(x)dx$ is finite. Therefore, if the integral $\int_1^\infty f(x)dx \to \infty$ then $\int_2^\infty f(x)dx \to \infty$, and therefore $\sum_{j=1}^\infty f(j+u) \to \infty$.

Suppose that $\int_1^\infty (-1)^x f(x)dx \to L$. It then follows $\forall \epsilon > 0, \exists M > 0$ such that

$$\left| \int_1^y (-1)^x f(x)dx - L \right| < \frac{\epsilon}{\pi\sqrt{2}} \quad y \geq M.$$

Thus, for $M \leq a \leq b$ it follows by the triangle inequality that

$$\left| \int_a^b (-1)^x f(x)dx \right| \leq \left| \int_1^a (-1)^x f(x)dx - L \right| + \left| \int_1^b (-1)^x f(x)dx - L \right| < \frac{\epsilon\sqrt{2}}{\pi}.$$

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Since $f(x)$ is decreasing, positive, and finite $\forall x \in \mathbb{R}^+$, it follows that f must converge to some value, say c, as $x \to \infty$. Since $c \leq f(x)$, it follows for $x \in [2n, 2n+1/2]$ with $n \in \mathbb{N}$ that

$$-\cos(\pi x) f(x) \leq c\cos(\pi x) \leq \cos(\pi x) f(x)$$

from which follows that

$$\left| \int_{2n}^{2n+1/2} c\cos(\pi x)dx \right| \leq \left| \int_{2n}^{2n+1/2} \cos(\pi x) f(x)dx \right|.$$

Knowing that

$$\int_{2n}^{2n+1/2} \cos(\pi x)dx = \frac{1}{\pi}$$

it follows that

$$\frac{c}{\pi} \leq \left| \int_{2n}^{2n+1/2} \cos(\pi x) f(x)dx \right|.$$

A similar argument will show that the same is true for the sine function. From this, it follows that

$$\frac{\sqrt{2}c}{\pi} \leq \left| \int_{2n}^{2n+1/2} (-1)^x f(x)dx \right|.$$

Let $n \geq M$. Then it follows that

$$\frac{\sqrt{2}c}{\pi} \leq \left| \int_{2n}^{2n+1/2} (-1)^x f(x)dx \right| < \frac{\epsilon\sqrt{2}}{\pi} \quad n \geq M$$

and therefore $c < \epsilon$. Therefore $c = 0$. Thus, $f(x) \to 0$ as $x \to \infty$, and therefore $f(u+j) \to 0, \forall u \in [0,1]$ as $j \to \infty$. From this, one can conclude that the series $\sum_{j=1}^{\infty}(-1)^j f(j+u)$ is convergent. Thus, if $\int_1^{\infty}(-1)^x f(x)dx$ is conditionally convergent, then $\sum_{j=1}^{\infty}(-1)^j f(j+u)$ is conditionally convergent $\forall u \in [0,1]$. □

Lemma 1. *Let f be a positive, decreasing, integrable function on every finite interval of $\mathbb{R}^+$. Then*

$$\int_0^1 f(u+j)\cos(\pi u)du \geq 0$$

for $\forall j \in \mathbb{N}$.

Proof. Note that

$$\begin{aligned} \int_0^1 f(u+j)\cos\pi u du &= \int_0^{1/2} f(u+j)\cos\pi u du + \int_{1/2}^1 f(u+j)\cos\pi u du \\ &\geq \int_0^{1/2} f(1/2+j)\cos\pi u du + \int_{1/2}^1 f(1/2+j)\cos\pi u du \\ &= f(1/2+j)\int_0^1 \cos\pi u du \\ &= 0. \end{aligned}$$

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□

Lemma 2. *Let $f(x)$ be positive, decreasing, and integrable on every finite subinterval of $\mathbb{R}^+$ such that*

- $\forall u \in [0,1]$ *and* $\forall j \in \mathbb{N}$, $f(u+j) - f(u+j+1) \leq f(j) - f(j+1)$;
- $\forall u \in [0,1]$ *the series* $\sum_{j=0}^{\infty}(-1)^j f(j+u)$ *is convergent.*

Then $(-1)^u \sum_{j=k}^{\infty}(-1)^j f(j+u)$ *is integrable on* $[0,1]$ *for each positive* k*, and*

$$\sum_{j=1}^{\infty}\int_0^1 (-1)^{u+j} f(u+j)du = \int_0^1 \sum_{j=1}^{\infty}(-1)^{u+j} f(u+j)du.$$

Proof. First to show integrability. Since $\sum_{j=k}^{\infty}(-1)^j f(j+u)$ is convergent it follows that $f(j+u) \to 0$ as $j \to \infty$. Thus, from the proof of Theorem 1 $\forall n,k \in \mathbb{N}$ with $n \geq k$,

$$\left|\sum_{j=k}^{n}(-1)^j f(u+j)\right| \leq \left|\sum_{j=k}^{n}(-1)^j f(j)\right|.$$

It follows from the hypothesis that f is integrable on $[j, j+1]$, and therefore $f(u+j)$ is integrable for $u \in [0,1]$. Fix k. Define

$$h_n(u) = \sum_{j=k}^{n}(-1)^j f(u+j).$$

Since the finite sum of integrable functions is integrable, $h_n(u)$ is integrable for each n. Therefore $\forall u \in [0,1]$, there exists a finite $h(u) \in \mathbb{R}$ such that $\lim_{n\to\infty} h_n(u) = h(u)$. Also $|h_n(u)| \leq |h_n(0)|$. Choose

$$M = \sup\{|h_k(0)|, |h_{k+1}(0)|, \ldots\}.$$

Since $h_n(0)$ is convergent and always finite, such a number exists. Therefore, $|h_n(u)| \leq M$ for $\forall u \in [0,1]$ and $\forall n \geq k$. Therefore, by Lebesgue Dominated Convergence Theorem, $h(u)$ is integrable on $[0,1]$.

Note that $(-1)^u h(u) = \cos(\pi u)h(u) + i\sin(\pi u)h(u)$, and that $\cos(\pi u)$ and $\sin(\pi u)$ are integrable $[0,1]$. Since the product of two integrable functions integrable, it follows that $\cos(\pi u)h(u)$ and $\sin(\pi u)h(u)$ are integrable $[0,1]$, and therefore $(-1)^u h(u)$ is integrable on $[0,1]$.

Now, in order to conserve space, define $g(x) = (-1)^x f(x)$. Thus, for any finite k

$$\sum_{j=1}^{\infty}\int_0^1 g(u+j)du = \sum_{j=1}^{k}\int_0^1 g(u+j)du + \sum_{j=k+1}^{\infty}\int_0^1 g(u+j)du$$

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and

$$\begin{aligned}\int_0^1 \sum_{j=1}^{\infty} g(u+j)du &= \int_0^1 \sum_{j=1}^{k} g(u+j)du + \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du \\ &= \sum_{j=1}^{k} \int_0^1 g(u+j)du + \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du.\end{aligned}$$

Therefore

$$\left|\sum_{j=k+1}^{\infty} \int_0^1 g(u+j)du - \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du\right| = \left|\sum_{j=1}^{\infty} \int_0^1 g(u+j)du - \int_0^1 \sum_{j=1}^{\infty} g(u+j)du\right|$$

for any finite positive k.

Let $u \in [0,1]$. Since f is decreasing and positive, it follows that $0 < f(u+j) - f(u+j+1)$, and thus,

$$0 < f(u+j) - f(u+j+1) \le f(j) - f(j+1).$$

Therefore, by Theorem 1

$$\left|\sum_{j=k+1}^{\infty} (-1)^j f(u+j)\right| \le \left|\sum_{j=k+1}^{\infty} g(j)\right|.$$

Thus,

$$-\sum_{j=k+1}^{\infty} g(j) \le \sum_{j=k+1}^{\infty} (-1)^j f(j+u) \le \sum_{j=k+1}^{\infty} g(j)$$

from which follows

$$\left|\int_0^1 \cos(\pi u) \sum_{j=k+1}^{\infty} (-1)^j f(u+j)du\right| \le \left|\int_0^1 \cos(\pi u) \sum_{j=k+1}^{\infty} g(j)du\right|$$

with the integrability of the left-hand side being given by Lemma 2. Evaluating the right-hand integral,

$$\left|\int_0^1 \cos(\pi u) \sum_{j=k+1}^{\infty} (-1)^j f(u+j)du\right| \le \frac{1}{\pi}\left|\sum_{j=k+1}^{\infty} g(j)du\right|.$$

The same follows identically for the sine function in place of the cosine function. Therefore, by the Triangle Inequality,

$$\left|\int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du\right| \le \frac{\sqrt{2}}{\pi}\left|\sum_{j=k+1}^{\infty} g(j)du\right|.$$

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Now, since $|\cos \pi u| \leq 1$, it follows that

$$-f(j) \leq -f(u+j) \leq f(u+j)\cos(\pi u) \leq f(u+j) \leq f(j).$$

Integrating u over $[0,1]$ yields

$$-f(j) \leq \int_0^1 f(u+j)\cos(\pi u)du \leq f(j).$$

Similarly, $|\int_0^1 f(u+j)\sin(\pi u)du| \leq f(j)$. By a similar argument,

$$-[f(j)-f(j+1)] \leq [f(u+j)-f(u+j+1)]\cos(\pi u) \leq f(j)-f(j+1)$$

and therefore

$$-[f(j)-f(j+1)] \leq \int_0^1 [f(u+j)-f(u+j+1)]\cos(\pi u)du \leq f(j)-f(j+1)$$

with similar following for the sine function. Note, for $u \in [0,1], 0 \leq \sin(\pi u) \leq 1$, and therefore $0 \leq [f(u+j)-f(u+j+1)]\sin(\pi u)$. Thus,

$$0 \leq \int_0^1 [f(u+j)-f(u+j+1)]\sin(\pi u)du.$$

By Lemma 1, for $\forall j \in \mathbb{N}$,

$$0 \leq \int_0^1 f(u+j)\cos(\pi u)du.$$

For every $u \in [0,1]$ and $y \in \omega$

$$[f(u+j)-f(u+j+1)]\cos \pi u \leq f(u+j)-f(u+j+1) \leq f(j)-f(j+1)$$

. Therefore

$$\int_0^1 [f(u+j)-f(u+j+1)]\cos \pi u du \leq \int_0^1 [f(j)-f(j+1)]du = f(j)-f(j+1)$$

or

$$\int_0^1 f(u+j)\cos \pi u du - \int_0^1 f(u+j+1)\cos \pi u du \leq f(j)-f(j+1)$$

. Letting $a_j = \int_0^1 f(u+j)\cos \pi u du$ and $b_j = f(j)$, it follows from Theorem 1 that

$$\left|\sum_{j=k+1}^{\infty} (-1)^j \int_0^1 f(u+j)\cos \pi u du\right| \leq \left|\sum_{j=k+1}^{\infty} (-1)^j f(j)\right|$$

for all k. Similarly,

$$\left|\sum_{j=k+1}^{\infty} (-1)^j \int_0^1 f(u+j)\sin \pi u du\right| \leq \left|\sum_{j=k+1}^{\infty} (-1)^j f(j)\right|$$

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By the Triangle Inequality

$$\left|\sum_{j=k+1}^{\infty}\int_0^1 g(u+j)du\right| \le \sqrt{2}\left|\sum_{j=k+1}^{\infty} g(j)\right|$$

and, again by the Triangle Inequality,

$$\left|\sum_{j=k+1}^{\infty}\int_0^1 g(u+j)du - \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du\right| \le \sqrt{2}\left(1+\frac{1}{\pi}\right)\left|\sum_{j=k+1}^{\infty} g(j)\right|.$$

Since $\sum_{j=1}^{\infty} g(j)$ is convergent, it follows that it is Cauchy, and therefore $\forall\epsilon > 0, \exists N \in \mathbb{N}$ such that

$$\left|\sum_{j=k+1}^{\infty} g(j)\right| < \frac{\epsilon}{\sqrt{2}\left(1+\frac{1}{\pi}\right)} \quad k \ge N.$$

Thus

$$\left|\sum_{j=k+1}^{\infty}\int_0^1 g(u+j)du - \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du\right| < \epsilon \quad k \ge N$$

which implies

$$\left|\sum_{j=1}^{\infty}\int_0^1 g(u+j)du - \int_0^1 \sum_{j=1}^{\infty} g(u+j)du\right| < \epsilon$$

and therefore

$$\sum_{j=1}^{\infty}\int_0^1 g(u+j)du = \int_0^1 \sum_{j=1}^{\infty} g(u+j)du.$$

□

2 Main Theorem

Theorem 3 (Main Theorem)**.** *Let f be a positive, decreasing function integrable on any finite subinterval of $\mathbb{R}^+$. If*

- $\forall u \in [0,1]$ *and* $\forall j \in \mathbb{N}$ *it holds that* $f(u+j) - f(u+j+1) \le f(j) - f(j+1)$
- $\int_1^{\infty}(-1)^x f(x)dx$ *is conditionally convergent*

then $\forall z \in \mathbb{C}$ *there exists a rearrangement on* $\int_1^{\infty}(-1)^x f(x)dx$ *such that ,* $z = \int_1^{\infty}(-1)^x f(x)dx$.

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Proof. Note
$$\int_1^\infty (-1)^x f(x)dx = \sum_{j=1}^\infty \int_j^{j+1} (-1)^x f(x)dx.$$
Making the substitution that $u = j + x$, it follows that
$$\int_1^\infty (-1)^x f(x)dx = \sum_{j=1}^\infty \int_0^1 (-1)^{u+j} f(u+j)du.$$
By Theorem 2, $\forall u \in [0,1]$ it holds that $\sum_{j=1}^\infty (-1)^j f(j+u)$ is conditionally convergent. Therefore, by Lemma 2
$$\int_1^\infty (-1)^x f(x)dx = \int_0^1 \sum_{j=1}^\infty (-1)^{u+j} f(u+j)du.$$
Since $\int_{j=1}^\infty f(u+j)(-1)^j$ is conditionally convergent $\forall u \in [0,1]$ it follows that $\forall h(u) \in \mathbb{R}$, there exists a rearrangement of the terms in the series such that $h(u) = \sum_{j=1}^\infty f(u+j)(-1)^j$. This constitutes a rearrangement on $\int_1^\infty (-1)^x f(x)dx$.

Choose $z \in \mathbb{C}$ such that $z = |z|e^{i\theta}$.

Now choose a rearrangement of the terms in $\sum_{j=1}^\infty f(u+j)(-1)^j$ for $\forall u \in [0,1]$ such that
$$\sum_{j=1}^\infty f(u+j)(-1)^j = 2|z|\cos(\pi u - \theta).$$
Then
$$\int_1^\infty (-1)^x f(x)dx = z.$$
□

Corollary 1. *Let f be a positive function, integrable on any finite subinterval of $\mathbb{R}^+$, with an everywhere negative and increasing derivative. If $\int_1^\infty (-1)^x f(x)dx$ is conditionally convergent, then $\forall z \in \mathbb{C}$, there exists a rearrangement on $\int_1^\infty f(x)dx$ such that $z = \int_1^\infty (-1)^x f(x)dx$.*

Proof. It suffices to show that f satisfies the requirements of the Main Theorem. The requirement of conditional convergence is obviously met. It is also clear that f is decreasing.

Since f is continuous, it follows by the Mean Value Theorem that there exists $c_j \in (j, j+u)$ such that
$$f(u+j) - f(j) = f'(c_j)u.$$
Since $(j, j+u) \cap (j+1, j+u+1) = \emptyset$, it follows that $c_j < c_{j+1}$, and $f'(c_j) < f'(c_{j+1})$. Thus,
$$f(u+j) - f(j) < f(u+j+1) - f(j+1)$$

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and

$$f(u+j) - f(u+j+1) < f(j) - f(j+1).$$

Thus, f satisfies the requirements of the Main Theorem. □

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