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UNIQUENESS OF SOLUTIONS IMPLIES EXISTENCE AND UNIQUENESS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper we are concerned with uniqueness implies uniqueness and uniqueness implies existence questions for solutions of a class of boundary value problems for the third order ordinary differential equation (ODE). First we show uniqueness of solutions of a class of two-point problems implies the uniqueness of solutions of an associated class of three-point problems. Then we establish uniqueness of solutions of the class of two-point problems implies the existence of solutions of the class of two point problems and the associated class of three-point problems.

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AMS (MOS) Subject Classifications: 34B15, 34B10

1 Introduction

In this paper, we are concerned with uniqueness and existence of solutions for a class of boundary value problems for the third order ordinary differential equation,

$$y'''(x) = f(x, y, y', y''), \quad a < x < b. \tag{1}$$

In particular, given $a < x_1 < x_2 < x_3 < b$ and $y_i \in \mathbb{R}$, $i = 1, 2, 3$, we are concerned with uniqueness implies uniqueness and uniqueness implies existence questions for solutions of (1) satisfying boundary conditions of the type

$$y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y'(x_2) = y_3, \tag{2}$$

$$y'(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = y_3, \tag{3}$$

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_3) = y_3, \tag{4}$$

or

$$y'(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3. \tag{5}$$

Questions of the types with which we deal are well studied for solutions of (1) satisfying conjugate type conditions; specifically, these are boundary conditions of the form

$$y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y''(x_1) = y_3, \tag{6}$$

$$y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y(x_2) = y_3, \tag{7}$$

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = y_3, \tag{8}$$

or

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3, \tag{9}$$

where $a < x_1 < x_2 < x_3 < b$ and $y_i \in \mathbb{R}$, $i = 1, 2, 3$.

The conditions (6) denote initial conditions. We shall refer to the conditions (7) or (8) as two-point conjugate conditions. We shall refer to the conditions (9) as three-point conjugate conditions.

We consider the two questions: **(i)** whether uniqueness of solutions of (1), (3) ((1), (2)) implies uniqueness of solutions of (1), (5) ((1), (4)) and **(ii)** whether uniqueness of solutions of (1), (3) ((1), (2)) implies existence of solutions of (1), (3) ((1), (2)) and whether uniqueness of solutions of (1), (3) ((1), (2)) implies existence of solutions of (1), (5) ((1), (4)). If both questions are answered in the affirmative, then uniqueness of solutions of (1), (3) ((1), (2)) implies existence of solutions of (1), (3) ((1), (2)) and uniqueness of solutions of (1), (3) ((1), (2)) implies existence and uniqueness of solutions of (1), (5) ((1), (4)).

Hypothesis 1.1. *With respect to equation* (1)*, we assume throughout that*

(A) $f(t, s_1, s_2, s_3) : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) *solutions of initial problems for* (1) *are unique and extend to* (a, b).

Jackson [4, 5], Hartman [2, 3], Lasota and Opial [6] and others have developed an analogous theory for the family of conjugate boundary value problems. In particular, it is known that uniqueness of solutions of (1), (7) implies existence and uniqueness of solutions of (1), (8) and of (1), (9). Likewise, uniqueness of solutions of (1), (8) implies existence and uniqueness of solutions of (1), (7) and of (1), (9).

We shall employ shooting methods and model our arguments after the work found [1].

2 Preliminary results

We shall be in need of two fundamental results. The first result, referred to as the Schrader precompactness condition [7], essentially provides criteria for sequential compactness.

Theorem 2.1. *Assume that solutions of* (1), (8) *are unique, when they exist. If* $\{y_n(x)\}$ *is a sequence of solutions of* (1) *which is uniformly bounded on a nondegenerate subinterval* $[c,d] \subset (a,b)$, *then there is a subsequence* $\{y_{n_l}(x)\}$ *such that* $\{y_{n_l}^{(i)}(x)\}$ *converges uniformly on each compact subinterval of* (a,b), $i = 0,1,2$.

The second result of this section gives continuous dependence of solutions of (1) with respect to the types of boundary conditions we consider. The proof applies a variation of the Brouwer invariance of domain theorem [8, p. 199] and we refer the reader to [4, 5] for the method of proof.

Theorem 2.2 (Brouwer Theorem)**.** *If* U *is an open subset of* $\mathbb{R}^n$, $\phi : U \to \mathbb{R}^n$ *is one-to-one and continuous on* U, *then* ϕ *is a homeomorphism and* $\phi(U)$ *is an open subset of* $\mathbb{R}^n$.

Theorem 2.3. *Assume that solutions of the boundary value problem* (1), (3) *are unique, when they exist. Given a solution* $y(x)$ *of* (1), *an interval* $[c,d] \subset (a,b)$, *points* $c < x_1 < x_2 < d$ *and an* $\epsilon > 0$, *there exists* $\delta(\epsilon, [c,d]) > 0$ *such that, if* $|x_i - t_i| < \delta$, $i = 1,2$ *and* $c < t_1 < t_2 < d$, *and if*

$$|y'(x_1) - z_1| < \delta, \quad |y(x_2) - z_2| < \delta, \quad |y'(x_2) - z_3| < \delta,$$

then there exists a solution $z(x)$ *of* (1) *satisfying*

$$z'(t_1) = z_1, \quad z(t_2) = z_2, \quad z'(t_2) = z_3,$$

and $|y^{(i)}(x) - z^{(i)}(x)| < \epsilon$ *on* $[c,d]$, $0 \le i \le 2$.

To apply the Brouwer invariance of domain theorem to obtain Theorem 2.3, let $x_0 \in (c,d)$ be fixed. Define $V = \{(x_1,x_2) : c < x_1 < x_2 < d\}$ and $U = V \times \mathbb{R}^3$ which is an open subset of $\mathbb{R}^5$. Define $\phi : U \times \mathbb{R}^5 \to \mathbb{R}^5$ by $\phi(x_1,x_2,c_1,c_2,c_3) = (x_1,x_2,y'(x_1),y(x_2),y'(x_2))$ where y is the unique solution of the IVP (1) satisfying the initial conditions

$$y(x_0) = c_1, \quad y'(x_0) = c_2, \quad y''(x_0) = c_3.$$

Theorem 2.3 is stated for the specific boundary conditions, (3). Each family of boundary conditions, (2)-(9), generates an analogous continuous dependence on parameters theorem.

3 Uniqueness of Solutions

Under Hypothesis 1.1, we establish in this section that uniqueness of solutions of (1), (3), implies uniqueness of solutions of (1), (5). We first establish via a theorem and its corollary that uniqueness of solutions of (1), (3) yields existence and uniqueness of solutions of each of the conjugate problems (1), (7), (1), (8), or (1), (9).

Theorem 3.1. *If solutions of* (1), (3) *are unique, when they exist, then solutions of* (1), (8) *are unique.*

Proof. Assume for the sake of contradiction that there exist two distinct solutions, y, z of (1) satisfying the same boundary conditions (8) for some $a < x_1 < x_2 < b$; that is, assume

$$y(x_1) = z(x_1), \quad y(x_2) = z(x_2) = y_2, \quad y'(x_2) = z'(x_2).$$

Set $w = y - z$. Apply Rolle's Theorem to w and find $s \in (x_1, x_2)$ such that $w'(s) = 0$. Then y and z are distinct solutions of (1) satisfying:

$$y'(s) = z'(s), \quad y(x_2) = z(x_2), \quad y'(x_2) = z(x_2)$$

□

In view of the uniqueness implies existence theory for the conjugate boundary value problems, we have an immediate corollary concerning existence and uniqueness of solutions of each of the conjugate type boundary value problems.

Corollary 3.1. *Assume that solutions of* (1), (3) *are unique, when they exist. Then each boundary value problem,* (1), (7), (1), (8), *or* (1), (9) *has a unique solution on* (a, b).

We now establish the main result of this section. We establish that uniqueness of solutions of (1), (3) implies uniqueness of solutions of (1), (5).

Theorem 3.2. *Assume that solutions of* (1), (3) *are unique, when they exist. Then solutions of* (1), (5) *are unique, when they exist.*

Proof. Assume for the sake of contradiction that there exist two distinct solutions, y, z of (1) satisfying the boundary conditions (5) for some $a < x_1 < x_2 < x_3 < b$, $y_i \in \mathbb{R}$, $i = 1, 2, 3$; that is, assume

$$y'(x_1) = z'(x_1),$$
$$y(x_2) = z(x_2),$$
$$y(x_3) = z(x_3),$$

where $x_1 < x_2 < x_3$.

Set $w = y - z$. We first show that the roots of w at x_2 and x_3 are simple; that is, $y'(x_i) \neq z'(x_i)$, $i = 2, 3$. Assume for the sake of contradiction that $0 = w'(x_i) = y'(x_i) - z'(x_i), i = 2$ or 3.

$$y'(x_1) = z'(x_1), \quad y(x_i) = z(x_i), \quad y'(x_i) = z'(x_i),$$

$i = 2$ or 3. Thus y and z are solutions of (1) satisfying the boundary conditions (3) for some $a < x_1 < x_i < b$. Since the solutions of (1) and (3) problems are unique, $y \equiv z$. This is a contradiction; hence

$$y'(x_i) \neq z'(x_i).$$

It is also the case that if $a < x < x_2$, $x_2 < x < x_3$, or $x_3 < x < b$, then $y(x) \neq z(x)$. If $y(x) = z(x)$, then y and z are distinct solutions of (1) satisfying three-point conjugate type boundary conditions at x, x_2, x_3. Assume without loss of generality that

$$y(x) > z(x), \quad a < x < x_2,$$

$$y(x) < z(x), \quad x_2 < x < x_3,$$

$$y(x) > z(x), \quad x_3 < x < b.$$

Next, let $n \geq 0$, and let y_n denote the solution of (1) satisfying the initial conditions.

$$y_n(x_1) = y(x_1), \quad y_n'(x_1) = y_1 = z'(x_1), \quad y_n''(x_1) = y''(x_1) + n. \tag{10}$$

Note that $y(x) = y_0(x)$. We shall show by contradiction that, for $0 \leq r < s$,

$$y(x) \leq y_r(x) < y_s(x) \quad x \in (x_1, b). \tag{11}$$

First assume $0 < r < s$. We shall show $y(x) < y_r(x)$ on (x_1, b). Assume there exists $x \in (x_1, b)$ such that $y_r(x) \leq y(x)$. Then there exists $\hat{x} \in (x_1, x]$ such that $y_r(\hat{x}) = y(\hat{x})$. In particular,

$$y_r(x_1) = y(x_1),$$
$$y_r'(x_1) = y'(x_1),$$
$$y_r(\hat{x}) = y(\hat{x}).$$

This contradiction violates the uniqueness of solutions of the two-point conjugate problem. Therefore, $y(x) < y_r(x)$ on (x_1, b). The proof that $y_r(x) < y_s(x)$ is completely analogous and omitted.

Allow us to define

$$E_0 = [x_2, x_3].$$

This will allow us to analyze the space where $r \geq 0$ and

$$E_r = \{x_2 \leq x \leq x_3 : y_r(x) \leq z(x)\},$$

and we shall show that $\{E_r\}$ is a family of compact, nested sets.

First, we will show that the family is nested. Let $0 < r < s$. It follows from (11) that if $0 < r < s$, then $E_s \subseteq E_r \subseteq (x_2, x_3)$. If $E_s \neq \emptyset$, we show

$$E_s \subset E_r \subset (x_2, x_3).$$

If $x_2 \in E_r$, then y_r and y are distinct solutions of a two-point conjugate problem

$$y_r(x_1) = y(x_1)$$

$$y_r'(x_1) = y'(x_1)$$

$$y_r(x_2) = y(x_2).$$

So $x_2 \notin E_r$. Similarly, $x_3 \notin E_r$. Thus $E_r \subset (x_2, x_3)$. If $x \in E_s$, then $y_r(x) < y_s(x) \leq z(x)$ and $x \in E_r$. So $E_s \subseteq E_r$. To show the set containment is strict, note by continuity, there exists $\hat{x} \in E_r$ such that $z(\hat{x}) = y_r(\hat{x}) < y_s(\hat{x})$. Then,

$$\hat{x} \in E_r, \hat{x} \notin E_s.$$

We now show each E_r is compact. $E_r \subset [x_2, x_3]$ and so, E_r is bounded. Let x be a limit point of E_r. Let $x_k \in E_r, k = 1, 2, ...$ and $x_k \to x$; then, by continuity, $y_r(x_k) \to y_r(x), z(x_k) \to z(x)$ and $y_r(x_k) \leq z(x_k)$. Thus $y_r(x) \leq z(x)$. Thus, we have shown $\{E_r\}$ is a family of compact, nested sets.

Also, we need to show that if E_r contains at least two distinct points, then E_r is an interval. Let E_r contain at least two distinct points and set

$$t_1 = \min\{t : t \in E_r\}$$

$$t_2 = \max\{t : t \in E_r\}.$$

By continuity, it follows that $y_r(t_1) = z(t_1)$, and $y_r(t_2) = z(t_2)$. As in the proof that $x_i \notin E_r$, $i = 2, 3$, if we set $w = y_r - z$, then $w'(t_1) < 0$ and $w'(t_2) > 0$. So, if there exists $x \in (t_1, t_2)$ such that $y_r(x) > z(x)$, then uniqueness of solutions of the three-point conjugate problem is violated. Therefore, E_r is an interval.

We claim that $E_r \neq \emptyset$, for every $r \geq 0$. For the sake of contradiction, assume for some $r > 0$, $E_r = \emptyset$; set

$$\eta = \sup_{\{0<s<r\}} \{s : E_s \neq \emptyset\}.$$

We argue that $E_\eta \neq \emptyset$. Assume for the sake of contradiction that $E_\eta = \emptyset$. Then

$$\min_{x \in [x_2, x_3]} |y_\eta(x) - z(x)| = \epsilon > 0.$$

Then, by continuous dependence on parameters for solutions of IVPs, there exists $\delta > 0$ such that if $|\eta - s| < \delta$, then

$$\max_{x \in [x_2, x_3]} |y_\eta(x) - y_s(x)| < \epsilon/2.$$

Thus if $\eta - \delta < s < \eta$, then $y_\eta(x) > y_s(x) > z(x)$ on $[x_2, x_3]$ and so $E_s = \emptyset$. This contradicts the definition of η and so, $E_\eta \neq \emptyset$.

Next, note that $y_\eta(x) = z(x)$, for every $x \in E_\eta$; for if there exists $x_0 \in E_\eta \to y_\eta(x_0) < z(x_0)$, then again by continuous dependence on parameters for IVPs, it follows that there exists $s > \eta$ such that

$$y_\eta(x_0) < y_s(x_0) < z(x_0).$$

This again contradicts the definition of η as a supremum. Hence, $y_\eta(x) = z(x)$, for every $x \in E_\eta$. By uniqueness of solutions of the three-point conjugate problem (9), E_η contains at most two distinct points, $\alpha_1 < \alpha_2$. By uniqueness of solutions of the two-point conjugate problems, it follows that neither $y'(\alpha_1) = z'(\alpha_1)$ nor $y'_\eta(\alpha_2) = z'(\alpha_2)$. As a consequence, there exists $x_0 \in (\alpha_1, \alpha_2)$ such that $y_\eta(x_0) < z(x_0)$, but we have shown that to be impossible. Hence it follows that E_η is a singleton, $E_\eta = \{\tau\}$. But then $y_\eta(\tau) = z(\tau)$ and $y_\eta(x) > z(x)$ on $[x_1, \tau) \cup (\tau, x_2]$, so that $y'_\eta(\tau) = z'(\tau)$; this contradicts uniqueness of solutions of (1), (3).

Since $E_\eta = \emptyset$ for some $r > 0$ leads to conclude that $E_r \neq \emptyset$, for every $r \geq 0$. In particular, for $n \in N, E_{n+1} \subset E_n \neq \emptyset$, and $E = \cap_{n=1}^{\infty} E_n \neq \emptyset$. Thus, $E \neq \emptyset$.

Next, we observe that E consists of a single point $x_1 < x_0 < x_2$. If $t_1, t_2 \in E$, with $x_2 < t_1 < t_2 < x_3$, then $t_1, t_2 \in E_r$. Hence, $[t_1, t_2] \subseteq \cap_{n=1}^{\infty} E_r = E$. The foregoing sets E_r are not null leads to the conclusion that the interval $[t_1, t_2] \subseteq E$. However, this implies that the sequence $\{y_n(x)\}$ is uniformly bounded on $[t_1, t_2]$; by the compactness condition, Theorem 2.1, a subsequence $\{y_{n_j}\}$ exists such that each $\{y_{n_j}^{(i)}\}$ converges uniformly, $i = 0, 1, 2$, and in particular, $\{y''_{n_j}(x_1)\}$ converges, which is a contradiction. Thus we conclude that $E = \{x_0\}$, with $x_2 < x_0 < x_3$, and

$$\lim_{n \to \infty} y_n(x_0) = y_0 \leq z(x_0)$$

and $y_n(x_0) \uparrow y_0 \leq z(x_0)$.

Now, we claim that this, too, is not possible. There are two cases to resolve.

(i) First, assume $y_0 = z(x_0)$. Then, for $\epsilon > 0$, sufficiently small, there is a solution $z(x, \epsilon)$ of a two-point conjugate problem, satisfying

$$z(x_0, \epsilon) = z(x_0) - \epsilon,$$

$$z(x_1, \epsilon) = z(x_1),$$

$$z'(x_1, \epsilon) = z'(x_1),$$

and $z(x, \epsilon) < z(x)$ on (x_1, x_3). Choose $\epsilon > 0$ such that,

$$y(x_0) < z(x_0, \epsilon) < z(x_0).$$

Let us note that $z'(x_1, \epsilon) = z'(x_1) = y'(x_1)$. Such a solution $z(x, \epsilon)$, can be used in place of z to construct a new collection of sets $\{E_n(\epsilon)\}$ with respect to the very same sequences of solutions, $\{y_n(x)\}$. Since $y_0 = z(x_0) > z(x_0, \epsilon)(x_0)$, we would obtain that $E(\epsilon) = \cap\{E_n(\epsilon)\} = \emptyset$. That leads to a contradiction, so $y_0 = z(x_0)$ is not possible.

(ii) Next, let us assume $y(x_0) < y_0 < z(x_0)$. Let $x_k \to x_0$. Construct a subsequence, $\{y_{n_k}\}$ of $\{y_n\}$, satisfying $y_{n_k}(x_k) \geq z(x_k)$. Since, $\lim_{n\to\infty} y_n(x_0) = y_0$, then

$$\lim_{k\to\infty} y_{n_k}(x_k) = y_0 \geq z(x_0).$$

But this contradicts that $y_0 < z(x_0)$.

From this final contradiction, we find that $y_0 \leq z(x_0)$ is not possible, and therefore solutions of (1), (5) are unique if they exist.

4 Existence of Solutions

Theorem 4.1. *Assume that solutions of* (1), (3) *are unique, when they exist. Then solutions of* (1), (3) *exist.*

Proof. Let $a < x_1 < x_2 < b, y_i \in \mathbb{R}, i = 1, 2, 3$. We shall show that there exists a solution, y, of (1) satisfying

$$y'(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = y_3.$$

There exists a unique solution $z(x)$ of (1) satisfying the two-point conjugate boundary conditions (7),

$$z(x_1) = 0,$$
$$z'(x_1) = y_1,$$
$$z(x_2) = y_2.$$

Next, we define the set

$$S : \{u'(x_2) : \text{u solution of (1)}, u'(x_1) = y_1, u(x_2) = y_2\} \subset \mathbb{R}.$$

Note that $z'(x_2) \in S$; thus, $S \neq \emptyset$.

Next, we will exhibit that S is a closed subset of $\mathbb{R}$. Assume for the sake of contradiction that S is not closed. Then there exists $r_0 \in \bar{S} \setminus S$ and a strictly monotone sequence $\{r_n\} \subset S$ such that $\lim_{n\to\infty} r_n = r_0$. Then, assume without loss of generality, that $r_n \uparrow r_0$. By definition of S, we signify that there exists a u_n that assures the solutions of (1) satisfying

$$u_n'(x_1) = y_1,$$
$$u_n(x_2) = y_2,$$
$$u_n'(x_2) = r_n.$$

So, essentially, because $r_n \in S, u_n$ exists. Note that $u_{n+1} - u_n$ satisfies boundary conditions

$$(u_{n+1} - u_n)'(x_1) = 0,$$

$$(u_{n+1} - u_n)(x_2) = 0,$$

$$(u_{n+1} - u_n)'(x_2) = r_{n+1} - r_n > 0,$$

where (1),(3), are unique solutions due to Theorem 2.3. Also, the uniqueness of the solutions of (5) provides,

$$(u_{n+1} - u_n)(x) < 0, \quad x \in (x_1, x_2), \tag{12}$$

$$(u_{n+1} - u_n)(x) > 0, \quad x \in (x_2, x_b). \tag{13}$$

To prove (12) and (13), let's assume for the sake of contradiction that $(u_{n+1} - u_n)(x) = 0$, where $x \in (x_1, x_b)$. If $(u_{n+1} - u_n)(x) = 0$ then this would be equivalent to (5). Consequently,

$$(u_{n+1} - u_n)(x) \neq 0$$

because previously, we verified that (5) has a unique solution. Hence, (12) and (13) hold.

Let v denote the solution of (1) satisfying the two-point conjugate conditions

$$v(x_1) = 0,$$

$$v(x_2) = y_2,$$

$$v'(x_2) = r_0.$$

Next, we want to show that there exist N_0, $x_1 < \tau_1 < x_2 < \tau_2 < b$, such that

$$u_{N_0}(\tau_1) = v(\tau_1),$$

$$u_{N_0}(x_2) = v(x_2),$$

$$u_{N_0}(\tau_2) = v(\tau_2).$$

Therefore $u_{N_0} \equiv v$ by uniqueness of solutions of three-point conjugate problems. But $u'_{N_0}(x_2) < v'(x_2)$. Thus, r_0 does not exist and S is closed.

To show the existence of N_0, τ_1, τ_2, note that by Theorem 2.1, u_n becomes unbounded on compact domains. If not, then a subsequence of u_n (and its derivatives) converges uniformly to a solution, w, of (1). Then $w'(x_2) = r_0$ since $u'_n(x_2) \to r_0$. In particular, $r_0 \in S$ which is a contradiction.

Thus, u_n becomes unbounded on compact domains. By (12), $u_n \uparrow$ to the right of x_2 and by (13), $u_n \downarrow$ to the left of x_2. $v'(x_2) > u'_n(x_2)$ for each n. So N_0, $x_1 < \tau_1 < x_2 < \tau_2 < b$ exist.

This completes the argument that S is closed.

Now we show S is open. Let $r \in S$. Apply Theorem 2.3; since the solution of (1) satisfying $y'(x_1) = y_1, y(x_2) = y_2, y'(x_2) = r$ is unique, then there exists $\delta > 0$ such that if $s \in (r-\delta, r+\delta)$, then there is a solution z of (1) satisfying $z'(x_1) = y_1, z(x_2) = y_2, z'(x_2) = s$; in particular, $(r-\delta, r+\delta) \subset S$ and S is open.

This completes the argument that S is open. Thus, $S = \mathbb{R}$ and $y_3 \in S$.

Theorem 4.2. *Assume that solutions of* (1), (3) *are unique, when they exist. Then solutions of* (1), (5) *exist.*

Proof. Let $a < x_1 < x_2 < x_3 < b, y_i \in \mathbb{R}, i = 1, 2, 3$. We shall show that there exists a solution, y, of (1) satisfying

$$y'(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3.$$

There exists a unique solution $z(x)$ of (1) satisfying the three-point conjugate boundary conditions (5),

$$z'(x_1) = y_1$$
$$z(x_3) = y_3$$
$$z'(x_3) = 0.$$

Next, we define the set

$$S : \{u(x_2) : \text{u solution of (1)}, \ u'(x_1) = y_1, u(x_3) = y_3\} \subset \mathbb{R}.$$

Next, we will exhibit that S is a closed subset of $\mathbb{R}$. Assume for the sake of contradiction that S is not closed. Then there exists $r_0 \in \bar{S} \setminus S$ and a strictly monotone sequence $\{r_n\} \subset S$ such that $\lim_{n\to\infty} r_n = r_0$. Then, assume without loss of generality, that $r_n \uparrow r_0$. By definition of S, we signify that there exists a u_n that assures the solutions of (1) satisfying

$$u_n'(x_1) = y_1,$$
$$u_n(x_2) = r_n,$$
$$u_n(x_3) = y_3.$$

So, essentially, because $r_n \in S, u_n$ exists. Note that $u_{n+1} - u_n$ satisfies boundary conditions

$$(u_{n+1} - u_n)'(x_1) = 0,$$
$$(u_{n+1} - u_n)(x_2) = r_{n+1} - r_n > 0,$$
$$(u_{n+1} - u_n)(x_3) = 0.$$

The uniqueness of the solutions of (1), (5) provides,

$$(u_{n+1} - u_n)(x) < 0, \quad x \in (x_1, x_3), \tag{14}$$

$$(u_{n+1} - u_n)(x) > 0, \quad x \in (x_3, x_b). \tag{15}$$

To prove (14) and (15), let's assume for the sake of contradiction that $(u_{n+1} - u_n)(x) = 0$, where $x \in (x_1, x_b)$. If $(u_{n+1} - u_n)(x) = 0$ then u_{n+1}, u_n are distinct solution satisfying

$$u'_{n+1}(x_1 = u'_n(x_1),$$

$$u_{n+1}(x) = u_n(x),$$
$$u_{n+1}(x_3) = u_n(x_3),$$

which violates uniqueness of solutions of (1), (5) problems.

Let v denote the solution of (1) satisfying the two-point conjugate conditions

$$v(x_2) = r_0,$$

$$v'(x_2) = 0,$$
$$v(x_3) = y_3.$$

Next, we want to show that there exist N_0, $x_1 < \tau_1 < x_2 < \tau_2 < x_3 b$, such that

$$u_{N_0}(\tau_1) = v(\tau_1),$$

$$u_{N_0}(\tau_2) = v(\tau_2),$$
$$u_{N_0}(x_3) = v(x_3).$$

Therefore $u_{N_0} \equiv v$ by uniqueness of solutions of three-point conjugate problems. But $u_{N_0}(x_2) < v(x_2)$. Thus, r_0 does not exist and S is closed.

To show the existence of N_0, $x_1 < \tau_1 < x_2 < \tau_2 < x_3 < b$, note that by Theorem 2.1, u_n becomes unbounded on compact domains. If not, then a subsequence of u_n (and its derivatives) converges uniformly to a solution, w, of (1). Then $w(x_2) = r_0$ since $u_n(x_2) \to r_0$. In particular, $r_0 \in S$ which is a contradiction.

Thus, u_n becomes unbounded on compact domains. By (14), $u_n \uparrow$ to the right of x_2 and by (15), $u_n \uparrow$ to the left of x_2. $v(x_2) > u_n(x_2)$ for each n. So N_0, $x_1 < \tau_1 < x_2 < \tau_2 < x_3 < b$ exist.

This completes the argument that S is closed.

Now we show S is open. Let $r \in S$. Apply Theorem 2.3; since the solution of (1) satisfying $y'(x_1) = y_1, y(x_2) = r, y(x_3) = y_3$ is unique, then there exists $\delta > 0$ such that if $s \in (r - \delta, r + \delta)$, then there is a solution z of (1) satisfying $z'(x_1) = y_1, z(x_2) = s, z(x_3) = y_3$; in particular, $(r - \delta, r + \delta) \subset S$ and S is open.

This completes the argument that S is open. Thus, $S = \mathbb{R}$ and $y_2 \in S$.

5 The Boundary Conditions (2) and (4)

In this section we briefly address the boundary value problems (1), (2) and (1), (4). Since the boundary conditions (2) and (4) represent mirror images of (3), (5) respectively, the theorems and proofs of sections 3 and 4 carry over in a complete analogous manner. The remaining theorems, stated without proof, associated with the boundary conditions (3) and (5) are as follows:

Theorem 5.1. *Assume that solutions of* (1), (2) *are unique, when they exist. Then solutions of* (1), (4) *are unique, when they exist.*

Theorem 5.2. *Assume that solutions of* (1), (2) *are unique, when they exist. Then solutions of* (1), (2) *exist.*

Theorem 5.3. *Assume that solutions of* (1), (2) *are unique, when they exist. Then solutions of* (1), (4) *exist.*

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