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MATHEMATICS WITH ONLY RODS

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Abstract. We discuss in this expository paper the rod system used in ancient China based on the mathematical classic work of Sun Zi, with a focus on application to solving systems of linear equations. The mathematics involved is authentic and beautiful, and we believe it is also of interest from historical, cultural, and pedagogical perspectives.

Keywords: *Book of Sun Zi, base ten notation, systems of linear equations*

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1. Introduction

This paper is devoted to an understanding of a very small portion of a piece of ancient Chinese mathematical artwork. More precisely, we attempt to understand the rod system used in ancient China, from the book of Sun Zi. The nature of this paper is expository, historical, cultural and pedagogical. In fact, one needs no more than high school algebra to understand the mathematics discussed in this paper. While the mathematics in this paper does not go beyond high school algebra, to fully understand the insights into this mathematics provided by the ancient Chinese requires serious reflection.

The Chinese mathematical treatise *The mathematical classic of Sun Zi* (see [2] for a full English translation of this book) written about 3rd century AD, provides the basis for our discussion in this paper. This book of Sun Zi illustrates many original thoughts, attempts and tools, usually through solving concrete real world practical problems, on arithmetic and algebra as they developed in ancient China. The most striking and interesting aspect of this book, at least to us, is their ingenious and systematical uses of rods to solve various different types of problems.

The rod system of Sun Zi is a base ten system of notation which uses arrangements of rods to denote numbers instead of written symbols (see section 3 for more details.) The major difference between this rod system and the base ten system we use today is once a modification of a numeral is made there is no record of the previous configuration. Moreover, many of the algorithms introduced by Sun Zi are different from those in use today. Sun Zi's solution of the pheasants and rabbits problem (see Problem 2.1 for a statement) is the focus of the remainder of this paper. For more details on the history of the treatise and explanations on the methods used there, as well as a full English translation, we invite you to read the excellent book [2].

The paper is organized as follows. As a preparation of the rod system, in section 2 we introduce the pheasants and rabbits problem and provide two different solutions. In section

3, we discuss the Chinese rod system in some depth. In particular, we describe how the rod system is applied in solving the system of n linear equations with n unknowns. In the last section, we raise some questions and present some cultural and pedagogical reasons why use of rod systems would be valuable even today.

2. Main example and some remarks

We present the 31st problem in Chapter three of the book, the so-called *pheasants and rabbits in the same cage problem.*

First, we state the problem.

Problem 2.1. There are many pheasants and rabbits in a cage. From the top one counts 35 heads. From the bottom one counts 94 legs. How many pheasants and rabbits are there in the cage?

We give two "modern" solutions. The first is standard and the second avoids algebra. Both solutions contain the idea of elimination.

Solution 2.1. Let x be the number of pheasants and y be the number of rabbits. Since there are total 35 heads, we have $x + y = 35$. Since each pheasant has 2 legs and each rabbit has 4 legs, we have another equation $2x + 4y = 94$. By Guassian elimination (either x or y), we get $x = 23$ and $y = 12$.

Solution 2.2. Assume there are only pheasants in the cage, so the number of pheasants is 35. It implies that there are 70 legs. However, there are 94 legs total, so we still have 24 legs not being counted. It means that they all come from rabbits. Since each rabbit has 4 legs, which is twice as many legs as each pheasants, we divide 24 by 2 and the result is 12 rabbits. Therefore, we have $35 - 12 = 23$ pheasants.

Now, we invite the readers to pose and consider the following question:

Question 2.1. How did ancient Chinese solve this problem? More precisely, how could one use only rods to solve such a problem?

In fact, rods are the only tools for ancient Chinese to play with in order to provide a complete solution for this particular math problem and many others. We make a few remarks on what we consider to be important connections between the rods and mathematics involved in this problem. We will discuss the Chinese rod system in more details in section 3.

Remark 2.1. As one observes in the first solution above, to find the answer, it is only necessary to look at the coefficients and constants of the two equations. Namely, from the first equation, we have the array of numbers 1 (coefficient of x), 1 (coefficient of y), 35 (constant). Similarly, we have the array of numbers 2 (coefficient of x), 2 (coefficient of y), 94 (constant). Thus, the question is reduced to how to represent these two arrays of numbers and how to use them to solve the problem. However, before taking any further steps, it is important to note that the ancient Chinese have already realized that this could be done! In other words, the ancient Chinese were very close to the notation of a matrix.

Remark 2.2. Sun Zi uses rods to represent numbers in an efficient way similar to a place value system or an abacus. In particular, fewer than 94 rods are used to represent the number 94. We describe this system in more detail in section 3.

Remark 2.3. Assuming one knows high school algebra, then one could easily understand and manipulate the equations to get rid of one variable, in this case, x or y. We can write our steps down on a piece of paper and check each step to make sure the answer is correct. However, the numbers, such as positive and negative fractions and their corresponding notations were not well-understood at the time. The paper itself, as a tool for people to write things on, were not available at the time Sun Zi's book was written. Moreover, at each step, one had to memorize all its previous steps before moving to the next step, simply due to the fact that the rods in display will alter the configuration at each step. See section 3 and [2].

3. Rod system

In this section, we describe the rod system used in the book of Sun Zi. In particular, we focus on its application to solving a system of linear equations, as we discussed in section 2. To fully understand this section, we ask the readers to draw pictures and use actual rods for themselves.

Suppose we have a bundle of counting rods (for example, bamboo sticks) and a counting board to display the rods. First, we describe the rod numerals that are needed in our application. We also invite the reader to try to find his or her own solutions before looking at our solutions. By remark 2.2, we first need to answer:

Question 3.1. How do we display the numbers?

Solution 3.1. *Step 1, we create our own notations for numbers from 0 to 9.* For 0, there is no rod. For 1 to 5, we use the vertical rods. For example, to denote 3, we use 3 vertical rods. For 6 to 9, we use a horizontal rod to denote 5 on the top of vertical rods. For example, to denote 8, a horizontal rod is over 3 vertical rods.

Step 2, we use the notation invented in step 1 to denote numbers larger than 9. We simply alternate the vertical rods and the horizontal rods when they are in consecutive decimals. For example, to denote 123, we put 1 vertical rod to denote 100, then put 2 horizontal rods to denote 20 next to it, then put 3 vertical rods to denote 3 at the end.

Remark 3.1. For simplicity, we restrict ourselves to positive integers. In fact, we can also use rods to represent fractions and negative fractions. See [2].

The second question we must answer is:

Question 3.2. How do we perform arithmetic?

While there are algorithms for using rods in subtraction (see [2, 3.2]) and division (see [2, 3.4]), we give some details for addition and multiplication here. First, we give an example to show how to add two numbers using rods.

Example 3.1. Add 2637 to 186: Put 2637 in the first row and 186 in the second row. Calculate from left to right. First take away 1 rod (from 186) add to 6 (from 2637) to make 7. Next, use 2 rods from the 3 (from 2637) add to 8 (from 186) to make it 10, the 1 from 10 carries to 7, to make 8. Likewise, use 4 rods from the 7 (last digit from 2637) add to 6 to make it 10, the 1 from 10 carries to 1 to make 2. At last, there is nothing remaining in the place of 186, the number appears in the first row reads as 2823.

The multiplication is also carried out from left to right. We show it by multiplying 49 and 38. See [2, 3.3] for more examples and details. The reader should compare the computation with multiplying $(40+9)(30+8)$ by foiling.

Example 3.2. Multiply 49 by 38: Put 49 in the upper row and 38 in the lower row. The upper 4 calls the lower 3, which is 12, so put 1200 in the middle row. The upper 4 calls the lower 8, which is 32 (indeed, 320), the middle row becomes 1520 (or rather looks like 152 as rods). Put away 40 in the upper row and shift the lower numeral one place to the right. Now the upper 9 calls the lower 3, which is 27 (indeed, 270), gives 1790 (or rather 179 as rods). The upper 9 calls the lower 8, which is 72, gives 1860. Then remove the upper and lower rows and what is left is the 1860.

Now we are ready to answer the last question:

Question 3.3. How do we carry out the steps in solving the system of linear equations?

We use Problem 2.1 as an example to illustrate the algorithm, used in the book of Sun Zi. Recall that there are two equations in the example: $x+y=35$ and $2x+4y=94$. Alternatively, we can use matrix notation to write them as: $\begin{bmatrix} 1 & 1 & 35 \\ 2 & 4 & 94 \end{bmatrix}$. We call this matrix A. Now we present the solution of the Problem 2.1 using rod system. Note that the display of rods for this problem is $A^T = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 35 & 94 \end{bmatrix}$, that is, the transpose of the matrix A (with entries of rods representing numbers).

Solution 3.2. (1) Multiply the first column by the upper right number in the second column, which is 2. We get a matrix $B_1 = \begin{bmatrix} 2 & 2 \\ 2 & 4 \\ 70 & 94 \end{bmatrix}$. To multiply 35 by 2, as in Example 3.2, first multiply 3 (or rather 30) and 2, then multiply 5 by 2, add the resulting numbers.

(2) Subtract the first column from the second column. We get $B_2 = \begin{bmatrix} 0 & 2 \\ 2 & 4 \\ 24 & 94 \end{bmatrix}$. We explain how 94 minus 70 is done: Put 94 in the first row and 70 in the second row. Simply remove the same type (horizontal or vertical) and same number of rods from each digit. Recall the display of 9 and 7, after removing the rods, it is left with 2, then the 4 from upper row remains since there is no rods from the second row.

(3) Use B_2, we start with the first column, use division, we can solve for one unknown, then use the second column we can solve the other unknown. To solve the second unknown one would need to perform subtraction, multiplication and division. We leave the missing details to the readers.

We summarize the algorithm into a proposition:

Proposition 3.1. *The rod system applied to solving a given system of linear equations with integers entries is carried through in the same manner as Gaussian elimination and they both lead to the same result.*

We give a sketch of proof to this observation.

Proof. We denote the matrix of a given system as A. In the corresponding rod system, the matrix becomes A^T. One solves a system A using Guassian elimination by row reduction on A to get the reduced form B. In the rod system, one uses column reduction on A^T to get the reduce form C, if necessary. The matrices B^T and C are equivalent as they are both equivalent to A^T. ∎

4. Comments and questions

In section 3, we only considered the system (or matrix) with positive integers as entries. Since the rod numerals offer us fractions and negative numbers, we should also expect to solve a system with fractional coefficients. In fact, such examples have been studied and discussed in Sun Zi's book, see for example [2, 7.4].

In this paper, we have only touched upon one application of the rod system. However, the rod system is used throughout the book of Sun Zi. It would be interesting to know the limitations of the rods (or other tools) and how far it can teach us mathematics. So we ask:

Question 4.1. Are there any applications of the rod system to more advanced mathematics?

Although the book of Sun Zi is a problem book on arithmetic and algebra, it shares a common spirit with Euclid's Elements. In both cases, one can get far-reaching mathematics starting with only a very small number of tools. In the case of Elements, the tools are definitions and postulates and propositions, while in the book of Sun Zi, the tools are only the rods at hand. In view of some current research and literature, for example, the paper [1], we consider it would be valuable in helping young students learn and teachers teach them basic math, such as arithmetic and learning fractions, solving polynomial equations, and understanding better Diophantine equations through rods.

Question 4.2. Can we combine the idea of using limited tools like rods with some current technology (like computer games, so that young students will be able to play with it on their iPhones for example) to help us achieve some of the goals we mentioned above?

Based on the positive feedback from the first author's experience with this project, who is a first year college student in American Culture and English (ACE) program and also a presenter in the talk (for which this paper is based on), we believe the following pedagogical remark is also meaningful.

Remark 4.1. Reading a book such as the mathematical classic of Sun Zi, it will motivate the ACE students to learn math, history and English. For example, by translating and comparing texts in different languages, the ACE students will learn more vocabularies and expressions used in mathematics. In fact, the project is ongoing and will serve as a part of the program in a math history course offered at Miami University. We hope to bring more undergraduates to participate and enjoy the beauty of mathematics and culture!

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