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FORCED OSCILLATIONS OF A CLASS OF NONLINEAR DISPERSIVE WAVE EQUATIONS AND THEIR STABILITY*

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Abstract It has been observed in laboratory experiments that when nonlinear dispersive waves are forced periodically from one end of undisturbed stretch of the medium of propagation, the signal eventually becomes temporally periodic at each spatial point. The observation has been confirmed mathematically in the context of the damped Korteweg-de Vries (KdV) equation and the damped Benjamin-Bona-Mahony (BBM) equation. In this paper we intend to show the same results hold for the pure KdV equation (without the damping terms) posed on a finite domain. Consideration is given to the initial-boundary-value problem

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \qquad 0 < x < 1, \ t > 0, \\ u(0,t) = h(t), \qquad u(1,t) = 0, \qquad u_x(1,t) = 0, \quad t > 0. \end{cases} \tag{$*$}$$

It is shown that if the boundary forcing h is periodic with small ampitude, then the small amplitude solution u of $(*)$ becomes eventually time-periodic. Viewing $(*)$ (without the initial condition) as an infinite-dimensional dynamical system in the Hilbert space $L^2(0,1)$, we also demonstrate that for a given periodic boundary forcing with small amplitude, the system $(*)$ admits a (locally) unique limit cycle, or forced oscillation, which is locally exponentially stable. A list of open problems are included for the interested readers to conduct further investigations.

Key words Forced oscillation, stability, the BBM equation, the KdV equation, time-periodic solution.

1 Introduction

In this paper we consider an initial-boundary-value problem (IBVP) of the Korteweg-de Vries (KdV) posed on the finite interval $(0,1)$, namely,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \qquad 0 < x < 1, \ t > 0, \\ u(0,t) = h(t), \quad u(1,t) = 0, \quad u_x(1,t) = 0, \qquad t > 0. \end{cases} \tag{1}$$

Guided by the outcome of laboratory experiments, interest is given to long time effect of the boundary forcing h and large time behavior of solutions of IBVP (1).

In the experiments of Bona, Pritchard and Scott[1], a channel partly filled with the water mounted with a flap-type wave maker at one end. Each experiment commenced with the water

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in the channel at rest. The wavemaker was activated and underwent periodic oscillations. The throw of the wavemaker and its frequency of oscillation was such that the surface waves brought into existence were of small amplitude and long wavelength, so it can be modeled by either Benjamin-Bona-Mahoney (BBM) type equation posed in a quarter plane:

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} - \alpha u_{xx} - \gamma u = 0, & x > 0,\ t > 0, \\ u(x,0) = 0, \qquad u(0,t) = h(t), & x \geq 0, \quad t \geq 0, \end{cases} \tag{2}$$

or the Korteweg-de Vries (KdV) type equation posed in a quarter plane:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_{xx} - \gamma u = 0, & x > 0,\ t > 0, \\ u(x,0) = 0, \qquad u(0,t) = h(t), & x \geq 0, \quad t \geq 0, \end{cases} \tag{3}$$

where α and γ are nonnegative constants that are proportional to the strength of the damping effect. The wavemaker is modeled by the boundary value function $h = h(t)$ which is assumed to be a periodic function of period τ.

It was observed in the experiments that at each fixed station down the channel, for example at a spatial point represented by x_0, that the wave motion $u(x_0, t)$, say, rapidly became periodic of the same period as the boundary forcing. This observation leads to the following conjecture for solutions of IBVPs (2) and (3).

Conjecture *If the boundary forcing h is a periodic function of period τ, then the solution u of IBVP* (2) *or IBVP* (3) *becomes eventually time-periodic of period τ, i.e.,*

$$\lim_{t\to\infty} \|u(\cdot, t+\tau) - u(\cdot, t)\|_{L^2(R^+)} = 0.$$

Furthermore, the following mathematical questions arise naturally which are important from the point of view of dynamical systems.

Questions *Assume the boundary forcing h is a periodic function* (*of period τ*).

1) *Does the equation in* (2) *or* (3) *admit a time periodic solution $u(x,t)$ of periodic τ satisfying the boundary condition? Such a time-periodic solution, if exists, is usually called a forced oscillation.*
2) *If such a time periodic solution exists, how many are there?*
3) *What are their their stability if those forced oscillations exist?*

In [3], Bona, Sun and Zhang studied the KdV type equation (3). Assuming the damping coefficient $\gamma > 0$, they showed that if the the amplitude of the boundary forcing h is small, then the solution u of (3) is asymptotically time-periodic satisfying

$$\|u(\cdot, t+\tau) - u(\cdot, t)\|_{L^2(R^+)} \leq Ce^{-\beta t} \qquad \text{for any } t \geq 0,$$

where C and β are two positive constants. In addition, they demonstrated that the equation in (3) admits a unique time-periodic solution $u^*(x,t)$ of period τ satisfying the boundary condition, which is shown to be globally exponentially stable in the space $H^s(R^+)$ with $s \geq 1$, i.e., for a given initial data $\phi \in H^s(R^+)$, the unique solution $u(x,t)$ of (3) with zero initial data replaced by $\phi(x)$ has the property

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{H^s(R^+)} \leq Ce^{-\beta t} \qquad \text{for any } t \geq 0,$$

where $C > 0$ is a constant depending only on $\|\phi\|_{H^s(R^+)}$. Later, the BBM type equation (3) was studied by Yang and Zhang[18]. The similar results have been established while assuming that both damping coefficients α and γ are greater than 0.

Earlier, the same problems had been studied by Zhang for the BBM equation[19] and the KdV equation[20] posed on the finite interval $(0, 1)$:

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} - \alpha u_{xx} - \gamma u = 0, \qquad 0 < x < 1,\ t > 0, \\ u(x, 0) = 0, \qquad 0 \le x \le 1, \\ u(0, t) = h(t), \qquad u(1, t) = 0 \quad t \ge 0, \end{cases} \tag{4}$$

and

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_{xx} - \gamma u = 0, \qquad x > 0,\ t > 0, \\ u(x, 0) = 0, \qquad 0 \le x \le 1, \\ u(0, t) = h(t), \qquad u(1, t) = 0, \qquad u_x(1, t) = 0, \quad t \ge 0. \end{cases} \tag{5}$$

Assuming either $\alpha > 0$ or $\gamma > 0$, Zhang[19,20] showed that if the boundary forcing h is a periodic function of period τ with small amplitude, then both solutions of the IBVPs (4) and (5) are asymptotically time periodic (of period τ). Moreover, both equations admit a unique time periodic solution satisfying the boundary conditions which is globally exponentially stable.

In the above cited works, they all require that the damping coefficients α and γ are greater than zero, or at least one of them is greater than zero. It would be interesting to see whether the results reported in those works still hold when both damping coefficients α and γ are zeros. This leads us to consider the IBVP (1) of the KdV equation posed on the finite interval $(0, 1)$. We will show, though there is no explicit damping term in the equation, the solution u of the IBVP (1) is asymptotically time-periodic if the boundary forcing h is periodic with small amplitude and the initial date ϕ is small in certain space. We will also demonstrate that for a given small amplitude, time-periodic boundary forcing h, the equation in (1) admits a (locally) unique time-periodic solution u^* satisfying the boundary conditions. Such a time-periodic solution is called forced oscillation or a limit cycle if we view (1) (without the initial condition) as an infinite-dimensional dynamical system in the Hilbert space $L^2(0, 1)$. This time-periodic solution will be shown to be locally exponential stable in the space $L^2(0, 1)$.

2 Preliminaries

In this section we first consider the following IBVP of the linear KdV equation with homogeneous boundary conditions:

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \qquad u(x, 0) = \phi(x), \qquad 0 < x < 1,\ t > 0, \\ u(0, t) = u(1, t) = u_x(1, t) = 0. \end{cases} \tag{6}$$

Its solution u can be written in the form

$$u(x, t) = W(t)\phi,$$

where $W(t)$ is the C^0-semigroup in the space $L^2(0, 1)$ generated by the operator

$$Af = -f' - f'''$$

with its domain $\mathcal{D}(A) = \{f \in H^3(0, 1);\ f(0) = f(1) = f'(1) = 0\}$. The operator A has the following spectral properties:

i) The spectrum $\sigma(A)$ of operator A consists of only countably many eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfying

$$Re\lambda_k \leq 0 \qquad \text{and} \qquad \lim_{k\to\infty} Re\lambda_k = -\infty;$$

ii) $i\mu \in \varrho(A)$, the resolvent set of A, for any $\mu \in R$;

iii) The eigenvalues, λ_k, $k = 1, 2, \cdots$, has the following asymptotic form

$$\lambda_k = -\frac{8\pi^3}{3\sqrt{3}}k^3 + O(k^2). \tag{7}$$

Moreover, the resolvent operator $R(\lambda, A)$ of A has the following asymptotic estimate as λ tends to ∞ along the pure imaginary axis:

$$\|R(i\omega, A)\| = O\big(|\omega|^{-\frac{2}{3}}\big), \qquad |\omega| \to \infty. \tag{8}$$

Consequently, $W(t)$ is a differential semigroup and there exists a $\beta > 0$ such that

$$\|W(t)\phi\|_{L^2(0,1)} \leq C\|\phi\|_{L^2(0,1)}e^{-\beta t}$$

for any $t \geq 0$, where $C > 0$ is a constant independent of ϕ and t. For given $T > 0$ and $t > 0$, let $Y_{t,T}$ be the space

$$Y_{t,T} = \big\{v \in C_b(t, t+T; L^2(0,1)) |\, v \in L^2(t, T+t; H^1(0,1))\big\},$$

which is a Banach space equipped with the norm

$$\|v\|_{Y_{t,T}} \equiv \sup_{t\leq t'\leq t+T} \|v(\cdot, t')\|_{L^2(0,1)} + \|v\|_{L^2(t,t+T;H^1(0,1))}.$$

Using the standard energy estimate method, one can show that the following estimate holds for solutions of IBVP (6).

Proposition 2.1 *Let $T > 0$ be given. For any $\phi \in L^2(0,1)$, the unique solution u of* (6) *belongs to the space $Y_{t,T}$ for any $t \geq 0$. Moreover, there exists constants $C > 0$ and $\beta > 0$ such that*

$$\|u\|_{Y_{t,T}} \leq C\|\phi\|_{L^2(0,1)}e^{-\beta t} \qquad \textit{for any } \ t > 0.$$

This result can be extended to the IBVP's linearized KdV equation with a variable coefficient:

$$\begin{cases} u_t + u_x + (au)_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \qquad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = u_x(1,t) = 0, \end{cases} \tag{9}$$

where $a = a(x,t)$ is a given function. For given $T > 0$, let Y_T be the space

$$Y_T = \left\{f(x,t) |\ f \in Y_{t,T} \text{ for any } t \geq 0;\ \sup_{t\geq 0} \|f\|_{Y_{t,T}} < +\infty\right\}.$$

The space Y_T is a Banach space equipped with the norm

$$\|f\|_{Y_T} \equiv \sup_{t\geq 0} \|f\|_{Y_{t,T}}.$$

The following proposition will play important roles in studying nonlinear KdV equation in the next section.

Proposition 2.2 *Let $W(t)$ be the semigroup introduced earlier and $T > 0$ be given such that $Q = W(T)$ satisfying $\|Q\| < 1$, where $\|Q\|$ denotes the operator norm of Q. Then there exist constants $\mu > 0$ and $\beta > 0$ such that if $a \in Y_T$ and $\|a\|_{Y_T} \leq \mu$, then (9) admits a unique solution $u \in Y_T$. Moreover*

$$\|u\|_{Y_{t,T}} \leq Ce^{-\beta t}\|\phi\|_{L^2(0,1)}$$

for any $t \geq 0$, where $C > 0$ is a constant independent of t and ϕ.

Finally we consider nonhomogeneous boundary value problem

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & u(x,0) = \phi(x), \\ u(0,t) = h(t), & u(1,t) = u_x(1,t) = 0. \end{cases} \tag{10}$$

The following estimate holds for its solutions.

Proposition 2.3 *There exist constant $C > 0$ and $\beta > 0$. If $h \in C_b^1(R^+)$, then*

$$\|u\|_{Y_{t,T}} \leq C\big(e^{-\beta t}\|\phi\|_{L^2(0,1)} + \|h\|_{C_b^1(R^+)}\big)$$

for any $t \geq 0$. If, in addition, there exists a $\delta_1 > 0$ such that $|h(t)| + |h'(t)| \leq Ce^{-\delta_1 t}$ for any $t \geq 0$, then

$$\|u\|_{Y_{t,T}} \leq C\left(e^{-\beta t}\|\phi\|_{L^2(0,1)} + e^{-\delta_1 t}\right)$$

for any $t \geq 0$.

3 Main Results

In this section we first consider long time behavior of solutions of the nonlinear system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & u(x,0) = \phi(x), \\ u(0,t) = h(t), & u(1,t) = u_x(1,t) = 0. \end{cases} \tag{11}$$

Here we assume that the boundary forcing $h \in C_b^1(R^+)$, but is not necessary a periodic function. The following result is an extension of Proposition 2.3.

Theorem 3.1 *Suppose that $T > 0$ is as given in Proposition 2.1. Then there exist $\delta_j > 0$, $j = 1, 2$ and $\beta > 0$ such that if $h \in C_b^1(R^+)$ satisfying*

$$\|\phi\|_{L^2(0,1)} + \|h\|_{C_b^1(R^+)} \leq \delta_1,$$

then the corresponding unique $u(x,t)$ of (11) satisfies

$$\|u\|_{Y_{t,T}} \leq C\big(\|\phi\|_{L^2(0,1)}e^{-\beta t} + \|h\|_{C_b^1(R^+)}\big)$$

where $C > 0$ is independent of ϕ and h. If, in addition,

$$\|h\|_{C_b^1(t,t+T)} \leq Ce^{-\delta_2 t}$$

for any $t \geq 0$, then

$$\|u\|_{Y_{t,T}} \leq C\left(\|\phi\|_{L^2(0,1)}e^{-\beta t} + e^{-\delta_2 t}\right) \tag{12}$$

for any $t \geq 0$.

Note that (12) implies that

$$\|u(\cdot,t)\|_{L^2(0,1))} \leq C_1 e^{-\beta^* t} \qquad \text{for any } t \geq 0,$$

where $\beta^* = \min\{\beta,\ \delta_1\}$.

Now we assume that h is a periodic function of period τ. For a given initial value $\phi \in L^2(0,1)$, let $u(x,t)$ is the corresponding solution and $w(x,t) = u(x,t+\tau) - u(x,t)$. Then $w(x,t)$ solves the following IBVP:

$$\begin{cases} w_t + w_x + w_{xxx} + (a(x,t)w)_x = 0, & w(x,0) = \phi^*(x), \\ w(0,t) = w(1,t) = w_x(1,t) = 0 \end{cases} \tag{13}$$

where $a(x,t) = \frac{1}{2}(u(x,t+\tau) + u(x,t))$ and $\phi^*(x) = u(x,\tau) - \phi(x)$.

According to Theorem 3.1, for a given $\mu > 0$, there exists a $\delta > 0$ such that if $\|\phi\|_{L^2(0,1)} + \|h\|_{C^1(0,\tau)} \le \delta$, then $a \in Y_T$ and $\|a\|_{Y_T} \le \mu$. The following theorem, which provides an affirmative answer for Problem 1 listed in the introduction, then follows from Proposition 2.2.

Theorem 3.2 *Assume that $h \in C_b^1(R^+)$ is periodic function of period τ. There exist $\beta > 0$ and $\delta > 0$ such that if $\|\phi\|_{L^2(0,1)} + \|h\|_{C^1(0,\tau)} \le \delta$, then the corresponding solution u of* (11) *satisfies*

$$\|u(\cdot,t+\tau) - u(\cdot,t)\|_{L^2(0,1)} \le Ce^{-\beta t} \qquad \text{for any } t \ge 0$$

where $C > 0$ is a constant depending only on δ.

Next our attention is turn to the pure boundary value problem

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \\ u(0,t) = h(t), \qquad u(1,t) = u_x(1,t) = 0. \end{cases} \tag{14}$$

The boundary forcing h is now assumed to be periodic with period τ. We are concerned with whether or not this periodic forcing generates a time-periodic solution of (14).

Theorem 3.3 *There exists a $\delta > 0$ such that if $h \in C_b^1(R^+)$ is a time-periodic function of period τ satisfying $\|h\|_{C^1(0,\tau)} \le \delta$, then* (14) *admits a solution*

$$u^* \in C_b(0,\infty; L^2(0,1)) \cap L^2_{loc}(0,\infty; H^1(0,1)),$$

which is a time-periodic function of period τ. In addition, there exist a $\eta > 0$ such that if $u_1^* \in C_b(0,\infty; L^2(0,1)) \cap L^2_{loc}(0,\infty; H^1(0,1))$ is a time-periodic solution of (14) *and*

$$\|u^*(\cdot,0) - u_1^*(\cdot,0)\|_{L^2(0,1)} \le \eta,$$

then $u^*(x,t) \equiv u_1^*(x,t)$ for any $x \in (0,1)$ and $t \ge 0$.

The Sketch of the Proof Choose $\phi \in L^2(0,1)$ and solve IBVP (11) to get the corresponding solution u. Let $w(x,t) = u(x,t+\tau) - u(x,t)$ for any $t \ge 0$. Then w solves IBVP (13). It can be shown that there exist $T > 0$, $\delta > 0$, and $\beta > 0$ such that if $\|\phi\|_{2(0,1)} + \|h\|_{C^1(0,\tau)} \le \delta$, then

$$\|w\|_{Y_{t,T}} \le Ce^{-\beta t}\|\phi^*\|_{L^2(0,1)} \qquad \text{for any } t \ge 0.$$

With this fact in hand, we can show that (14) possesses a time-periodic solution of period τ.

Let $u_n = u(x,n\tau)$ for $n \ge 1$. For any positive integers n and m,

$$\begin{aligned} \|u_{n+m} - u_n\|_{L^2(0,1)} &= \left\| \sum_{k=1}^m u_{n+k} - u_{n+k-1} \right\|_{L^2(0,1)} \le \sum_{k=1}^m \|w(\cdot,(n+k-1)\tau)\|_{L^2(0,1)} \\ &\le \frac{Ce^{-n\tau}}{1-e^{-\tau}}\|\phi^*\|_{L^2(0,1)} \end{aligned}$$

for any $m \geq 1$. Thus, $\{u_n\}$ is a Cauchy sequence in $L^2(0,1)$. Let $\psi \in L^2(0,1)$ be the limit of u_n as $n \to \infty$. By Theorem 3.1, $\|\psi\|_{L^2(0,1)} \leq C\delta$. Taking ψ as an initial data together with the boundary forcing h for IBVP (11), we claim that its solution u^* is the desired time-periodic solution of period τ.

Indeed, note that while $u_n(\cdot) = u(\cdot, n\tau)$ converges strongly to ψ and $u_{n+1}(\cdot) = u(\cdot, n\tau + \tau)$ converges strongly to $u^*(\cdot, \tau)$ in $L^2(0,1)$ as $n \to \infty$ because of the continuity of the associated solution map. Observing that

$$
\begin{aligned}
&\|u^*(\cdot,\tau) - u^*\cdot, 0)\|_{L^2(0,1)} \\
\leq\ &\|u^*(\cdot,\tau) - u^*(\cdot, n\tau + \tau)\|_{L^2(0,1)} + \|u^*(\cdot, n\tau+\tau) - u^*(\cdot, n\tau)\|_{L^2(0,1)} \\
&+\|u^*(\cdot, 0) - u^*(\cdot, n\tau)\|_{L^2(0,1)}
\end{aligned}
$$

for any $n \geq 1$, it is concluded that $u^*(\cdot,\tau) = u^*(\cdot, 0)$ in $L^2(0,1)$ and therefore u^* is a time-periodic function of period τ.

To show the uniqueness, let u_1^* be another time-periodic solution with the same boundary forcing. Let $z(x,t) = u^*(x,t) - u_1^*(x,t)$. Then z solves the IBVP (13) with $a = u_1^* + u^*$ and $\phi^*(x) = u_1^*(x,0) - \psi(x)$. One can show that z decays exponentially in the space $L^2(0,1)$ as $t \to \infty$. Thus $u_1^*(\cdot, t) = u^*(\cdot, t)$ in the space $L^2(0,1)$ for any $t \geq 0$. ∎

For a given periodic boundary forcing h of period τ, the IBVP (11) may be viewed as a dynamic system in the infinite dimensional space $L^2(0,1)$. It has been just proved that if the amplitude of h is small, the equation in (11) admits a time-periodic solution $u^*(x,t)$ of period τ satisfying the boundary conditions in (11). Maintaining the dynamic systems perspective, one may view u^* as a limit cycle of the dynamic system. A natural further inquiry is then to study the stability of this limit cycle.

Theorem 3.4 *Under the assumption of Theorem* 3.3, *the time-periodic solution u^* is locally exponentially stable in the space $L^2(0,1)$, which is to say, there exist $\theta > 0$, $\gamma > 0$, and $C > 0$ such that for any $\phi \in L^2(0,1)$ satisfying $\|\phi(\cdot) - u^*(\cdot, 0)\|_{L^2(0,1)} \leq \theta$, the corresponding solution u of* (11) *satisfies*

$$\|u(\cdot,t) - u^*(\cdot,t)\|_{L^2(0,1)} \leq Ce^{-\gamma t} \qquad \textit{for any } t \geq 0.$$

Proof Let $w(x,t) = u(x,t) - u^*(x,t)$. Then w solves the IBVP (13) with $a(x,t) = \frac{1}{2}(u(x,t) + u^*(x,t))$ and $\phi^*(x) = \phi(x) - u^*(x,0)$. Similar as before, one can show that

$$\|w(\cdot,t)\|_{L^2(0,1)} \leq C_1\|\phi^*\|_{L^2(0,1)}e^{-\gamma t}$$

for any $t \geq 0$ if $\phi(\cdot)$ is close enough to $u^*(\cdot,0)$ in the space $L^2(0,1)$. ∎

4 Conclusion Remarks

There have been many studies concerning with time-periodic solutions of partial differential equations in the literature. Early work on this subject include for example, Brézis[5], Vejvoda et. al[15], Keller and Ting[10] and Rabinowitz[13,14]. For recent theory, see Craig and Wayne[7] and Wayne[17]. In particular, the interested readers are referred to article [17] in which Wayne has provided a very helpful review of theory pertaining to time-periodic solutions of nonlinear partial differential equations.

In this paper, motivated by laboratory experiments, we have studied the initial-boundary-value problem of the KdV equation posed on the finite interval $(0,1)$:

$$
\begin{cases}
u_t + u_x + uu_x + u_{xxx} = 0, & 0 < x < 1,\ t > 0, \\
u(x,0) = \phi(x), \quad u(0,t) = h(t), \quad u(1,t) = u_x(1,t) = 0, & 0 < x < 1,\ t > 0.
\end{cases}
\tag{15}
$$

We have proved that if the boundary forcing is periodic of small amplitude, then any solution of (15) is asymptotically time periodic so long as the norm of its initial value ϕ in the space $L^2(0,1)$ is small. In addition, the equation in (15) admits a unique small amplitude time-periodic solution satisfying the boundary conditions in (15), which is locally exponentially stable in $L^2(0,1)$.

We conclude the paper with a list of open questions with remarks for the interested readers to conduct further investigations.

Open problems

1) The results presented in the paper for the IBVP (15) are local in the sense that both amplitudes of the initial value and the boundary forcing are required to be small. In particular, both uniqueness and existence of time-periodic solution as well as its stability are local in nature. Do those results hold globally? In particular, is the small amplitude time-periodic solution globally exponentially stable in the space $L^2(0,1)$? A more challenging question is whether a periodic boundary forcing of large amplitude produce a large amplitude time-periodic solution?
2) Study IBVP (4) of the BBM equation posed on the finite interval $(0,1)$ with both the damping coefficients $\alpha = 0$ and $\gamma = 0$, i.e.,

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, \qquad 0 < x < 1,\ t > 0, \\ u(x,0) = \phi(x), \quad u(0,t) = h(t), \quad u(1,t) = 0, \qquad 0 < x < 1,\ t > 0. \end{cases} \tag{16}$$

If the boundary forcing h is a periodic function, does the equation in (16) admit a time-periodic solution satisfying the boundary conditions. If such a time-periodic solution exists, how many are there and what are their stability?

The interested readers should be reminded that while BBM equation is usually easier to study mathematically than the KdV equation, it is the other way around, however, for the problems proposed here. This is because the system described by IBVP (16) of the BBM equation is conserved in the sense the H^1-norm of its solution is conserved:

$$\frac{d}{dt}\int_0^1 (u_t^2(x,t) + u_x^2(x,t))dx = 0 \qquad \text{for any } t \geq 0$$

if the boundary forcing $h \equiv 0$. By contrast, if the boundary forcing $h \equiv 0$, the small amplitude solution of IBVP (15) of the KdV equation (with $\alpha = 0$ and $\beta = 0$) decays exponentially to zero in the space $L^2(0,1)$. The system described by IBVP (15) possesses a built-in dissipative mechanism through imposing the boundary condition at $x = 0$ although there is no damping term appeared in the equation. A different approach is needed to study time-periodic solution of system (16).
3) Study the following IBVP of the KdV type equation posed on the half line R^+:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_x - \gamma u = 0, \qquad 0 < x,\ t < \infty, \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t), \qquad 0 < x,\ t < \infty, \end{cases} \tag{17}$$

where $\alpha \geq 0$ and $\gamma \geq 0$ are constants.

As we have pointed out in the introduction, Bona, Sun and Zhang[3] have shown that if the damping coefficient $\gamma > 0$ and the boundary forcing is periodic with small amplitude, then system (17) admits a unique amplitude time-periodic solution which is also globally exponentially stable in the space $H^s(R)$ with $s \geq 1$. In their proof, the condition $\gamma > 0$ is crucial. A question arises naturally what will happen if $\gamma = 0$? A more challenging case is when both damping coefficients α and γ are zero.

References

[1] J. L. Bona, W. G. Pritchard, and L. R. Scott, An evaluation of a model equation for water waves, *Philos. Trans. Royal Soc. London Series A*, 1981, **302**: 457–510.

[2] J. L. Bona, S. Sun, and B. Y. Zhang, A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane, *Trans. American Math. Soc.*, 2001, **354**: 427–490.

[3] J. L. Bona, S. M. Sun, and B. Y. Zhang, Forced oscillations of a damped Korteweg-de Vries equation in a quarter plane, *Comm. Contemp. Math.*, 2003, **5**(3): 369–400.

[4] J. L. Bona, S. Sun, and B. Y. Zhang, A nonhomogeneous boundary value problem for the KdV equation posed on a finite domain, *Commun. Partial Differential Eq.*, 2003, **28**: 1391–1436.

[5] H. Brézis, Periodic solutions of nonlinear vibrating strings and duality principles, *Bull. Amer. Math. Soc.*, 1983, **8**: 409–425.

[6] P. Constantin, C. Foiaş, B. Nicolaenko, and R. Temam, Integral manifolds and inertial manifolds for dissipative partial differential equations, in *Applied Mathematical Sciences*, Vol. 70, Springer-Verlag, New York-Berlin, 1989.

[7] W. Craig and C. E. Wayne, Newton's method and periodic solutions of nonlinear wave equations, *Comm. Pure Appl. Math.*, 1993, **46**: 1409–1498.

[8] J. M. Ghidaglia, Weakly damped forced Korteweg-de Vries equations behave as a finite-dimensional dynamical system in the long time, *J. Differential Eqns.*, 1988, **74**: 369–390.

[9] J. M. Ghidaglia, A note on the strong convergence towards attractors of damped forced KdV equations, *J. Differential Eqns.*, 1994, **110**: 356–359.

[10] J. B. Keller and L. Ting, Periodic vibrations of systems governed by nonlinear partial differential equations, *Comm. Pure and Appl. Math.*, 1966, **19**(2): 371–420.

[11] J. U. Kim, Forced vibration of an aero-elastic plate, *J. Math. Anal. Appl.*, 1986, **113**: 454–467.

[12] G. R. Sell and Y. C. You, Inertial manifolds: the nonselfadjoint case, *J. Diff. Eqns.*, 1992, **96**: 203–255.

[13] P. H. Rabinowitz, Periodic solutions of nonlinear hyperbolic differential equations, *Comm. Pure Appl. Math.*, 1967, **20**: 145–205.

[14] P. H. Rabinowitz, Free vibrations for a semi-linear wave equation, *Comm. Pure Appl. Math.*, 1978, **31**: 31–68.

[15] O. Vejvoda, *Partial Differential Equations: Time-Periodic Solutions*, Mrtinus Nijhoff Publishers, 1982.

[16] C. E. Wayne, Periodic solutions of nonlinear partial differential equations, *Notices Amer. Math. Soc.*, 1997, **44**: 895–902.

[17] C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Comm. Math. Phys.*, 1990, **127**: 479–528.

[18] Y. Yang and B. Y. Zhang, Forced oscillations of a damped Benjamin-Bona-Mahony equation in a quarter plane, *Control Theory of Partial Differential Equations*, 375–386, Lect. Notes Pure Appl. Math., 242, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[19] B. Y. Zhang, Forced oscillations of a regularized long-wave equation and their global stability, *Differential Equations and Computational Simulations* (Chengdu, 1999), World Sci. Publishing, River Edge, NJ, 2000, 456–463.

[20] B. Y. Zhang, Forced oscillation of the Korteweg-de Vries-Burgers equation and its stability, *Control of Nonlinear Distributed Parameter Systems* (College Station, TX, 1999), Lecture Notes in Pure and Appl. Math., Vol. 218, Dekker, New York, 2001, 337–357.